3.6 The Chain Rule

Sometimes we need to take the derivative of an expression raised to a large power, and it would take too long to multiply all of it out and use the power rule. The chain rule allows us to be able to do this. We will start with $y = f(u)$. The outside function is f and the inside function will be u. The chain rule says the following:

Chain Rule

If $y = f(u)$ then $y' = f'(u)\cdot u'$ Which is also the same as $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$ just using different notation.

What you are really doing is taking the derivative of a outside function and multiplying it with the derivative of the inside function. Now let's look at some examples.

EXAMPLE: Write the expression in the form $y = f(u)$ and $u = g(x)$ Then find $\frac{dy}{dx}$ if $y = \frac{-5}{(x+3)^3}$.

First we can rewrite this as $y = -5(x+3)^{-3}$. For this problem, they want you to consider our function as a composition of two other functions. The inside function is the *u*. So, $u = x+3$. Now, if we replace the inside function with u, then we will get the expression for y: $y = -5u^{-3}$. Now we want to find the derivative. We will use the formula $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$. So $\frac{dy}{dx} = \frac{d}{du}(-5u^{-3})\cdot\frac{d}{dx}(x+3)$. This will give us: $\frac{dy}{dx} = 15u^{-4}\cdot 1$. We simplify and put the *u* back in to get: $\frac{dy}{dx} = 15(x+3)^{-4}$. This can be simplified to: $\frac{dy}{dx} = \frac{15}{(x+3)^4}$.

EXAMPLE: Write the expression in the form $y = f(u)$ and $u = g(x)$ Then find $\frac{dy}{dx}$ if $y = \left(\frac{1}{6x} - \frac{x}{6}\right)^6$.

We can rewrite this problem first: $y = \left(\frac{1}{6}x^{-1} - \frac{1}{6}x\right)^6$ For this problem, they want you to consider our function as a composition of two other functions. The inside function is the *u*. So, $u = \frac{1}{6}x^{-1} - \frac{1}{6}x$. Now, if we replace the inside function with *u*, then we will get the expression for y: $y = u^6$. Now we want to find the derivative. We will use the formula $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$. So $\frac{dy}{dx} = \frac{d}{du}(u^6)\cdot\frac{d}{dx}\left(\frac{1}{6}x^{-1} - \frac{1}{6}x\right)$. This will give us:

$\frac{dy}{dx} = 6u^5\cdot\left(-\frac{1}{6}x^{-2} - \frac{1}{6}\right)$. We simplify and put the *u* back in to get: $\frac{dy}{dx} = 6\left(\frac{1}{6x} - \frac{x}{6}\right)^5\left(-\frac{1}{6}x^{-2} - \frac{1}{6}\right)$. We can multiply the 6 into the second factor to get: $\frac{dy}{dx} = \left(\frac{1}{6x} - \frac{x}{6}\right)^5\left(-1 - \frac{1}{x^2}\right)$.

EXAMPLE: Write the expression in the form $y = f(u)$ and $u = g(x)$ Then find $\frac{dy}{dx}$ if $y = \csc(\cot\theta)$.

For this problem, they want you to consider our function as a composition of two other functions. The inside function is the u. So, $u = \cot\theta$. Now, if we replace the inside function with u, then we will get the expression for y: $y = \csc u$. Now we want to find the derivative. We will use the formula $\frac{dy}{d\theta} = \frac{dy}{du}\cdot\frac{du}{d\theta}$. So $\frac{dy}{d\theta} = \frac{d}{du}(\csc u)\cdot\frac{d}{d\theta}(\cot\theta)$. This will give us: $\frac{dy}{dx} = -\csc u\cot u \cdot -\csc^2\theta$. We simplify and put the u back in to get: $\frac{dy}{d\theta} = \csc(\cot\theta)\cot(\cot\theta)\csc^2\theta$.

EXAMPLE: Write the expression in the form $y = f(u)$ and $u = g(x)$ Then find $\frac{dy}{dx}$ if $y = (9x-7)^4$.

The inside function is the u. So, $u = 9x-7$. Now, if we replace the inside function with u, then we will get the expression for y: $y = u^4$. Now we want to find the derivative. We will use the formula $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$. So $\frac{dy}{dx} = \frac{d}{du}(u^4)\cdot\frac{d}{dx}(9x-7)$. This will give us: $\frac{dy}{dx} = 4u^3\cdot 9 = 36u^3$. Now we put the u back in to get: $\frac{dy}{dx} = 36(9x-7)^3$.

EXAMPLE: Write the expression in the form $y = f(u)$ and $u = g(x)$ Then find $\frac{dy}{dx}$ if $y = \sqrt{3x^2-4x+6}$.

The inside function is the u. So, $u = 3x^2-4x+6$. Now, if we replace the inside function with u, then we will get the expression for y: $y = \sqrt{u}$. Now we want to find the derivative. We will use the formula $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$. So $\frac{dy}{dx} = \frac{d}{du}(\sqrt{u})\cdot\frac{d}{dx}(3x^2-4x+6)$. This will give us: $\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}}(6x-4) = \frac{2(3x-2)}{2\sqrt{u}}$. Now we put the u back in and simplify to get: $\frac{dy}{dx} = \frac{3x-2}{\sqrt{3x^2-4x+6}}$.

EXAMPLE: Write the expression in the form $y = f(u)$ and $u = g(x)$ Then find $\frac{dy}{dx}$ if $y = \sin^5 x$.

This problem can be rewritten as $y = (\sin x)^5$ The inside function is the u. So, $u = \sin x$. Now, if we replace the inside function with u, then we will get the expression for y: $y = u^5$. Now we want to find the derivative. We will use the formula $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$. So $\frac{dy}{dx} = \frac{d}{du}(u^5)\cdot\frac{d}{dx}(\sin x)$. This will give us: $\frac{dy}{dx} = 5u^4\cos x$. Now we put the u back in and simplify to get: $\frac{dy}{dx} = 5\sin^4 x\cos x$.

EXAMPLE: Find the derivative of the function: $y = \sqrt[6]{x^7 - 5x^4}$.

The inside function is $u = x^7 - 5x^4$. The outside function (written with a rational exponent) would be $y = u^{\frac{1}{6}}$. The derivative is always the derivative of the outside function times the derivative of the inside function: $y' = \frac{1}{6}u^{-\frac{5}{6}} \cdot \left(7x^6 - 20x^3\right)$. Now we put in our u and rewrite it with positive exponents: $y' = \frac{7x^6 - 20x^3}{6\left(x^7 - 5x^4\right)^{\frac{5}{6}}}$.

EXAMPLE: Find y' if $y = x\sqrt{4 + x^2}$.

Sometimes we may need to combine either the product rule or quotient rule with the chain rule. In this problem we need to use the product rule and chain rule. First we will start with the product rule. The *f* is *x* and the *g* is $\sqrt{4 + x^2}$.

$$y' = \underbrace{x}_{f} \cdot \underbrace{\frac{1}{2}(4 + x^2)^{-\frac{1}{2}}(2x)}_{g'} + \underbrace{\sqrt{4 + x^2}}_{g}\underbrace{(1)}_{f'}$$

When we get to the part of the product rule where we need to find g', we need to use the chain rule with $u = 4 + x^2$ and $f(x) = x^{\frac{1}{2}}$.

$$g' = \underbrace{\frac{1}{2}(4 + x^2)^{-\frac{1}{2}}}_{f'(u)}\underbrace{(2x)}_{u'}.$$

This simplifies to $y' = \frac{x^2}{\sqrt{4 + x^2}} + \sqrt{4 + x^2}$. We can now get common denominators:

$y' = \frac{x^2}{\sqrt{4 + x^2}} + \frac{\sqrt{4 + x^2}}{1}\left(\frac{\sqrt{4 + x^2}}{\sqrt{4 + x^2}}\right)$. This simplifies to $y' = \frac{x^2 + 4 + x^2}{\sqrt{4 + x^2}} = \frac{2x^2 + 4}{\sqrt{4 + x^2}}$.

EXAMPLE: Find y' if $y = \frac{(6x + 1)^4}{(2x + 1)^5}$.

This problem combines the quotient rule with the chain rule. We will use the quotient rule and also the chain rule when we find the derivative of the top and bottom:

$$y' = \frac{\underbrace{(2x + 1)^5}_{g} \cdot \underbrace{4(6x + 1)^3(6)}_{f'} - \underbrace{(6x + 1)^4}_{f} \cdot \underbrace{5(2x + 1)^4(2)}_{g'}}{\left((2x + 1)^5\right)^2}.$$

$$y' = \frac{24(2x + 1)^5(6x + 1)^3 - 10(6x + 1)^4(2x + 1)^4}{(2x + 1)^{10}}$$

Now we want to factor out a common factor.

$$y' = \frac{(2x+1)^4(6x+1)^3\left[24(2x+1)-10(6x+1)\right]}{(2x+1)^{10}}$$ Now simplify as much as possible.

$$y' = \frac{(6x+1)^3\left[48x+24-60x-10\right]}{(2x+1)^6}$$

$$y' = \frac{(6x+1)^3\left[-12x+14\right]}{(2x+1)^6} = \frac{-2(6x-7)(6x+1)^3}{(2x+1)^6}$$

EXAMPLE: Find $h'(t)$ if $h(t) = \left(\frac{t^2}{t^3+2}\right)^2$. Write your answer as a single fraction.

This problem involves the chain rule combined with the quotient rule. The first part of this problem involves the power rule with the outside function. When you multiply this by the derivative of the inside the quotient will then be used.

$$h'(t) = 2\left(\frac{t^2}{t^3+2}\right)^1\left(\frac{(t^3+2)(2t)-(t^2)(3t^2)}{(t^3+2)^2}\right)$$ We can distribute and add on the top of the second fraction.

$$h'(t) = \frac{2t^2}{t^3+2}\cdot\frac{4t-t^4}{(t^3+2)^2}$$ Now multiply across the top and across the bottom

$$h'(t) = \frac{8t^3-2t^6}{(t^3+2)^3}$$ The last thing we can do is factor the top.

$$h'(t) = \frac{2t^3(4-t^3)}{(t^3+2)^3}$$ Nothing cancels, so this is our final answer.

EXAMPLE: Find $f'(\theta)$ if $f(\theta) = \sin(\cos(9\theta))$.

Sometimes you may need to use the chain rule more than once. Start with the outside most function, which is sine in this case. We will take the derivative of that first and we will get cosine. Then multiply that by the derivative of $\cos(9\theta)$. To do this derivative you need to use the chain rule again. The derivative of $\cos(9\theta)$ is $-\sin(9\theta)\cdot 9$. The extra 9 on the outside of that is the derivative of the inner most part, which is the derivative of 9θ. Putting that all together you will get $f'(\theta) = \cos(\cos(9\theta))\cdot -\sin(9\theta)\cdot 9$. After some simplifying you will get: $f'(\theta) = -9\cos(\cos(9\theta))\cdot\sin(9\theta)$.

EXAMPLE: Find y' if $y = \cos\left[(4x+7)^{\frac{3}{4}}\right]$.

Here is another one that uses the chain rule more than once. Start with the outside most function, which is cosine in this case. We will take the derivative of that first and we will get negative sine. Then multiply that by the derivative of $(4x+7)^{\frac{3}{4}}$. To do this derivative you need to use the chain rule again. The derivative of $(4x+7)^{\frac{3}{4}}$ is $\frac{3}{4}(4x+7)^{-\frac{1}{4}}(4)$. The extra 4 on the outside of that is the derivative of the inner most part, which is the derivative of $4x+7$. Putting that all together you will get $y' = -\sin\left[(4x+7)^{\frac{3}{4}}\right]\cdot\frac{3}{4}(4x+7)^{-\frac{1}{4}}(4)$. After simplifying you will get: $y' = -3\sin\left[(4x+7)^{\frac{3}{4}}\right]\cdot(4x+7)^{-\frac{1}{4}}$.

EXAMPLE: Find y' if $y = 2\cot^2(\pi x+2)$.

First I will rewrite this as $y = 2(\cot(\pi x+2))^2$. Like the previous problem start with the outside most function, which involves the power rule. Once the power rule is used then we must find the derivative of the inside part, which is the derivative of $\cot(\pi x+2)$ which involves the chain rule again. The derivative of this part is $-\csc^2(\pi x+2)(\pi)$. The extra π on the outside is the derivative of the inner most part, which is the derivative of $\pi x+2$. Putting this all together you will get: $y' = 4\cot(\pi x+2)(-\csc^2(\pi x+2))\cdot\pi$. Simplifying it you will get: $y' = -4\pi\cot((\pi x+2))\cdot\csc^2(\pi x+2)$

EXAMPLE: Find y' if $y = e^{-x^2}$.

For this problem we will use $\frac{d}{dx}\left[e^u\right] = e^u\cdot u'$. The derivative of e is itself multiplied by the derivative of the exponent. Here $u = -x^2$. Then $u' = -2x$. So our answer is: $y' = e^{-x^2}(-2x)$ which can be written as: $y' = -2x\cdot e^{-x^2}$.

EXAMPLE: Find y' if $y = \left(e^{\sin(x/2)}\right)^6$.

First use the power rule for the outside part, then we must find the derivative of the inside part, which is the derivative of $y = e^{\sin(x/2)}$ which involves the chain rule again. The derivative of this part is $e^{\sin(x/2)}\cdot\cos(x/2)\cdot(1/2)$. The extra $1/2$ on the outside is the derivative of the inner most part, which is the derivative of $x/2$. Putting this all together you will get: $y' = 6\left(e^{\sin(x/2)}\right)^5 e^{\sin(x/2)}\cdot\cos(x/2)\cdot(1/2)$. Simplifying it you will get: $y' = 3\cos\left(\frac{x}{2}\right)\left(e^{\sin(x/2)}\right)^6$. You could also write this as $y' = 3\cos\left(\frac{x}{2}\right)\left(e^{6\sin(x/2)}\right)$ using exponent rules.

EXAMPLE: Find the derivative of $y = \sin\left(\frac{\theta}{\sqrt{\theta - 4}}\right)$.

We first start with the outside. The derivative of sine is cosine. Then we need to take the derivative of the inside. The derivative of the inside function involves a quotient rule:

$$y' = \cos\left(\frac{\theta}{(\theta-4)^{\frac{1}{2}}}\right)\left(\frac{(\theta-4)^{1/2}(1) - \theta \cdot \frac{1}{2}(\theta-4)^{-1/2}}{\theta - 4}\right)$$ Notice I changed the original denominator to $(\theta-4)^{-\frac{1}{2}}$.

$$y' = \cos\left(\frac{\theta}{(\theta-4)^{\frac{1}{2}}}\right)\left(\frac{(\theta-4)^{1/2} - \frac{\theta(\theta-4)^{-1/2}}{2}}{\theta - 4}\right)$$ I will divide everything on top by $(\theta-4)$.

$$y' = \cos\left(\frac{\theta}{(\theta-4)^{\frac{1}{2}}}\right)\left(\frac{(\theta-4)^{1/2}}{\theta-4} - \frac{\theta(\theta-4)^{-1/2}}{2(\theta-4)}\right)$$ Now I use exponent rules to simplify.

$$y' = \cos\left(\frac{\theta}{(\theta-4)^{\frac{1}{2}}}\right)\left(\frac{1}{(\theta-4)^{1/2}} - \frac{\theta}{2(\theta-4)^{3/2}}\right)$$ I will get common denominators on top.

$$y' = \cos\left(\frac{\theta}{(\theta-4)^{\frac{1}{2}}}\right)\left(\frac{2(\theta-4)}{2(\theta-4)^{3/2}} - \frac{\theta}{2(\theta-4)^{3/2}}\right)$$ After simplifying I get: $y' = \cos\left(\frac{\theta}{(\theta-4)^{\frac{1}{2}}}\right)\left(\frac{(\theta-8)}{2(\theta-4)^{3/2}}\right)$

This is the final answer. I am not allowed to multiply inside the cosine. This remains unchanged.