3.8 Derivatives of Inverse and Logarithmic Functions

Derivatives of Inverses of Differentiable Functions

EXAMPLE: Given $f(x)=x^2-1$ and $f^{-1}(x)=\sqrt{x+1}$, find $\frac{d}{dx}f(x)$ and $\frac{d}{dx}f^{-1}(x)$.

We are going to find each derivative and see if we can make a connection between them. The derivatives are:

$\frac{d}{dx}f(x)=2x$ (Using Power Rule)

$\frac{d}{dx}f^{-1}(x)=\frac{d}{dx}\left((x+1)^{\frac{1}{2}}\right)=\frac{1}{2}(x+1)^{-\frac{1}{2}}(1)=\frac{1}{2\sqrt{x+1}}$ (Using Chain Rule)

Let's look at the denominator of $f^{-1}(x)$ which is $2\sqrt{x+1}$. The 2 comes from the derivative of *f*(x). The square root is from our inverse. So it looks like the inverse is put inside of the derivative of *f*. Therefore $2\sqrt{x+1}$ can be written generally as $f'\left(f^{-1}(x)\right)$. We can generalize this:

Derivative Rule for Inverses

$$(f^{-1})'(x)=\frac{1}{f'\left(f^{-1}(x)\right)}$$

EXAMPLE: Given $f(x)=\sqrt{x+7}$ and $f^{-1}(x)=x^2-7$, evaluate $\frac{d}{dx}f(x)$ at $x=2$ and $\frac{d}{dx}f^{-1}(x)$ at $x=f(2)$

Let's take derivatives of each function:

$$\frac{d}{dx}f(x)=\frac{d}{dx}(x+7)^{\frac{1}{2}}=\frac{1}{2}(x+7)^{-\frac{1}{2}}(1)=\frac{1}{2\sqrt{x+7}}$$

$$\frac{d}{dx}f^{-1}(x)=\frac{d}{dx}\left(x^2+7\right)=2x$$

Next we evaluate $\frac{d}{dx}f(x)$ at $x=2$: $\frac{d}{dx}f(2)=\frac{1}{2\sqrt{2+7}}=\frac{1}{6}$

The we evaluate $\frac{d}{dx}f^{-1}(x)$ at $x=f(2)=3$: $\frac{d}{dx}f^{-1}(3)=2(3)=6$

Notice that these are inverses of each other which should be expected.

EXAMPLE: Let $f(x)=3x^2-7x+2$. Find the value of $\frac{d}{dx}f^{-1}(x)$ at $x = 0 = f(2)$.

What we know about inverses is the x and y are switched. We are told that $f(2)=0$. From the concept of inverses we know $f^{-1}(0)=2$. If b is any number in the domain of the inverse, then $(f^{-1})'(b)=\frac{1}{f'\left(f^{-1}(b)\right)}$.

So in our problem our b will be 0. So $(f^{-1})'(0)=\frac{1}{f'\left(f^{-1}(0)\right)}$. Since $f^{-1}(0)=2$ we can substitute:

$(f^{-1})'(0)=\frac{1}{f'(2)}$. To find this we will first find the derivative of f: $f'(x)=6x-7$. We will evaluate this at $x = 2$: $f'(2)=6(2)-7=5$. So $(f^{-1})'(0)=\frac{1}{5}$. We were able to find this without finding a formula for the inverse of f.

Derivative of a Natural Logarithm

Let u be a differentiable function of x. Then:

1.) $\frac{d}{dx}[\ln x]=\frac{1}{x}$ where $x>0$

2.) $\frac{d}{dx}[\ln u]=\frac{1}{u}\cdot\frac{du}{dx}=\frac{u'}{u}$ where $u>0$

EXAMPLE: Find the derivative: $y=\ln\left(\frac{9}{x^2}\right)$.

So we will apply the second formula above, so $u=\frac{9}{x^2}$. To find u' we will first rewrite: $u=9x^{-2}$. So $u'=-18x^{-3}$. Our derivative is $y'=\frac{u'}{u}$, so $y'=\frac{-18x^{-3}}{9x^{-2}}$. Now we need to simplify: $y'=-2x^{-3-(-2)}$ so $y'=-\frac{2}{x}$.

EXAMPLE: Find the derivative: $f(x)=\ln|\csc x|$.

We have $u=\csc$ and $u'=-\csc x\cot x$. Our derivative is: $f'(x)=\frac{u'}{u}$, which is $f'(x)=\frac{-\csc x\cot x}{\csc x}$. This simplifies to: $f'(x)=-\cot x$.

EXAMPLE: Find the derivative: $y=\frac{1-\ln x}{x}$.

You will need to use the quotient rule on this one since it can't be simplified with log properties. You are also not allowed to cancel the x's on this one. We will use the quotient rule and when we get to the part of the problem where we take the derivative of $\ln x$ we will put $\frac{1}{x}$: $\frac{x\left(-\frac{1}{x}\right)-(1-\ln x)(1)}{x^2}$. Simplified: $\frac{-2+\ln x}{x^2}$.

EXAMPLE: Find the derivative: $y=\ln(\sin(\ln\theta))$.

This requires the use of the chain rule. First for the outside natural log, let $u=\sin(\ln\theta)$. Then when we find u', this will require us to use the chain rule. So $u'=\cos(\ln\theta)\cdot\frac{1}{\theta}$. There is $\frac{1}{\theta}$ on the end because this was the derivative of the inside function, $\ln\theta$. Now let's put it together: $y'=\frac{\cos(\ln\theta)/\theta}{\sin(\ln\theta)}$. This can be simplified to: $y'=\frac{\cos(\ln\theta)}{\theta\sin(\ln\theta)}$. Using a trig identity you could also write this as: $y'=\frac{\cot(\ln\theta)}{\theta}$.

EXAMPLE: Find the derivative: $y=\ln\left(\frac{2x}{x+3}\right)$.

For this one, at first you might pick $u=\frac{2x}{x+3}$ and then use the quotient rule to find u'. You could do this and you would get the correct answer. However we can make this problem easier by first using the log property #2: $y=\ln 2x-\ln(x+3)$. Now we can take the derivative of each term separately. In the first term we have $u=2x$ and $u'=2$. In the second term we have $u=x+3$ and $u'=1$. We will apply the formula $\frac{u'}{u}$ when taking the derivative of each log, so $y'=\frac{2}{2x}-\frac{1}{x+3}$. We can get common denominators: $y'=\frac{2(x+3)-2x}{2x(x+3)}$. The numerator simplifies: $y'=\frac{6}{2x(x+3)}$. We can reduce this to get our answer: $y'=\frac{3}{x(x+3)}$.

EXAMPLE: Find the derivative: $y=\ln\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}$.

Will use log property #4 to get rid of the exponent: $y=\frac{1}{3}\ln\left(\frac{x-1}{x+1}\right)$. Now we can use property #2 to break up the fraction: $y=\frac{1}{3}\left[\ln(x-1)-\ln(x+1)\right]$. This is also $y=\frac{1}{3}\ln(x-1)-\frac{1}{3}\ln(x+1)$. Now we can take the derivative of each term separately. In the first term we have $u=x-1$ and $u'=1$. In the second term we have $u=x+1$ and $u'=1$. Our derivative is $y'=\frac{u'}{u}$ which is: $y'=\frac{1}{3}\cdot\frac{1}{x-1}+\frac{1}{3}\cdot\frac{1}{x+1}$. We can get common denominators: $y'=\frac{x+1-(x-1)}{3(x-1)(x+1)}$, which simplifies to: $y'=\frac{2}{3(x-1)(x+1)}$.

EXAMPLE: Find the derivative: $y = \ln\sqrt{\dfrac{\sin\theta\cos\theta}{1-4\ln\theta}}$.

We first want to apply log properties to break this up:
$y = \frac{1}{2}\ln\left(\frac{\sin\theta\cos\theta}{1-4\ln\theta}\right) = \frac{1}{2}\left[\ln(\sin\theta\cos\theta) - \ln(1-4\ln\theta)\right] = \frac{1}{2}\ln\sin\theta + \frac{1}{2}\ln\cos\theta - \frac{1}{2}\ln(1-4\ln\theta)$.

Now we will take the derivative of each term separately. For each log we will use the formula $\dfrac{u'}{u}$:

$y' = \frac{1}{2}\cdot\frac{\cos\theta}{\sin\theta} + \frac{1}{2}\cdot\frac{-\sin\theta}{\cos\theta} - \frac{1}{2}\cdot\frac{-4\cdot\frac{1}{\theta}}{1-4\ln\theta}$. Now simplify: $y' = \frac{1}{2}\cot\theta - \frac{1}{2}\tan\theta + \frac{2}{\theta(1-4\ln\theta)}$

EXAMPLE: Find the derivative: $y = \ln\left(\dfrac{1+e^x}{1-e^x}\right)$.

You should use the log properties to first break this apart: $y = \ln(1+e^x) - \ln(1-e^x)$. Now take the derivative of each term separately. In the first term, $u = 1+e^x$ and $u' = e^x$. In the second term, $u = 1-e^x$ and $u' = -e^x$.

Putting it all together we have: $y' = \dfrac{e^x}{1+e^x} - \dfrac{-e^x}{1-e^x}$. We can get common denominators:

$y' = \frac{e^x}{1+e^x}\cdot\left(\frac{1-e^x}{1-e^x}\right) + \frac{e^x}{1-e^x}\cdot\left(\frac{1+e^x}{1+e^x}\right)$. Now simplify and combine as one fraction.

$y' = \frac{e^x(1-e^x) + e^x(1+e^x)}{(1+e^x)(1-e^x)}$ Multiply the top terms. Remember to add the exponents.

$y' = \frac{e^x - e^{2x} + e^x + e^{2x}}{(1+e^x)(1-e^x)}$ We can now simplify to get our answer.

$y' = \frac{2e^x}{(1+e^x)(1-e^x)}$

Logarithmic Differentiation

The process of logarithmic differentiation involves taking the derivative of both sides of the equation. This process is usually done to release a variable from the exponent position or it can be used to break up a product or a quotient. This process is easier than using the chain rule in combination with product and quotient rules.

EXAMPLE: Use logarithmic differentiation to find the derivative of: $y = \sqrt[3]{x(x-4)}$.

First we rewrite this as $y = (x(x-4))^{\frac{1}{3}}$, which is the same as $y = x^{\frac{1}{3}}(x-4)^{\frac{1}{3}}$. Now take the natural log of both sides: $\ln y = \ln\left(x^{\frac{1}{3}}(x-4)^{\frac{1}{3}}\right)$. Before taking the derivative of both sides, use log properties to break this up:

$\ln y = \ln x^{\frac{1}{3}} + \ln(x-4)^{\frac{1}{3}}$. Now bring down the powers: $\ln y = \frac{1}{3}\ln x + \frac{1}{3}\ln(x-4)$. We are now ready to take the derivative of both sides: $\frac{y'}{y} = \frac{1}{3}\cdot\frac{1}{x} + \frac{1}{3}\cdot\frac{1}{x-4}$. Simplify to get: $\frac{y'}{y} = \frac{1}{3x} + \frac{1}{3(x-4)}$. Now solve for y' by multiplying both sides of the equation by y. You will get $y' = y\left(\frac{1}{3x} + \frac{1}{3(x-4)}\right)$. We were given that $y = \sqrt[3]{x(x-4)}$, so substitute this in for y and get our final answer: $y' = \sqrt[3]{x(x-4)}\left(\frac{1}{3x} + \frac{1}{3(x-4)}\right)$. Notice this problem would be more difficult if we did it with the chain rule and product rule.

EXAMPLE: Use logarithmic differentiation to find the derivative of: $y = (x-2)^{x+1}$.

First take the natural log of both sides: $\ln y = \ln\left((x-2)^{x+1}\right)$. Then we can use log property #4 to bring down the exponent: $\ln y = (x+1)\ln(x-2)$. We are now ready to take the derivative of both sides. On the right side we will need to use the power rule with $f = x+1$ and $g = \ln(x-2)$: $\frac{y'}{y} = (x+1)\cdot\frac{1}{x-2} + \ln(x-2)(1)$. Simplify to get: $\frac{y'}{y} = \frac{x+1}{x-2} + \ln(x-2)$. Now solve for y' by multiplying both sides of the equation by y. You will get $y' = y\left(\frac{x+1}{x-2} + \ln(x-2)\right)$. Since $y = (x-2)^{x+1}$, our answer is: $y' = (x-2)^{x+1}\left(\frac{x+1}{x-2} + \ln(x-2)\right)$.

EXAMPLE: Use logarithmic differentiation to find the derivative of: $y = \frac{\theta\sin\theta}{\sqrt{\sec^3\theta}}$.

First we rewrite this as $y = \frac{\theta\sin\theta}{(\sec\theta)^{\frac{3}{2}}}$. This can also be written as: $y = \theta\sin\theta(\cos\theta)^{\frac{3}{2}}$. Now take the natural log of both sides: $\ln y = \ln\left[\theta\sin\theta(\cos\theta)^{\frac{3}{2}}\right]$. Before taking the derivative of both sides, use log properties to break this up: $\ln y = \ln(\theta) + \ln(\sin\theta) + \frac{3}{2}\ln(\cos\theta)$. We are now ready to take the derivative of both sides: $\frac{y'}{y} = \frac{1}{\theta} + \frac{\cos\theta}{\sin\theta} + \frac{3}{2}\cdot\frac{-\sin\theta}{\cos\vartheta}$. Simplify to get: $\frac{y'}{y} = \frac{1}{\theta} + \cot\theta - \frac{3}{2}\tan\theta$. Now solve for y' by multiplying both sides of the equation by y. You will get $y' = y\left(\frac{1}{\theta} + \cot\theta - \frac{3}{2}\tan\theta\right)$. We were given that $y = \frac{\theta\sin\theta}{\sqrt{\sec^3\theta}}$, so substitute this in for y and get our final answer: $y' = \frac{\theta\sin\theta}{\sqrt{\sec^3\theta}}\left(\frac{1}{\theta} + \cot\theta - \frac{3}{2}\tan\theta\right)$.

Derivative of a^x

To do this one we will first start with an identity: $a^x = e^{(\ln a)x}$. Now we will take the derivative of both sides:

$\frac{d}{dx}\left[a^x\right] = \frac{d}{dx}\left[e^{(\ln a)x}\right]$. The derivative of the right side involves the chain rule.

$\frac{d}{dx}\left[a^x\right] = e^{(\ln a)x} \cdot \ln a$ Now we can rewrite this to get the following result:

$\frac{d}{dx}\left[a^x\right] = (\ln a)a^x$ or $\frac{d}{dx}\left[a^u\right] = (\ln a)a^u \cdot u'$

Derivative of $\log_a x$

First we will rewrite the log as $\frac{\ln x}{\ln a} = \frac{1}{\ln a} \cdot \ln x$ by using the change of base formula for logs. Now take the derivative of both sides:

$\frac{d}{dx}\left[\log_a x\right] = \frac{d}{dx}\left[\frac{1}{\ln a} \cdot \ln x\right]$ Now we take the derivative of the right side.

$\frac{d}{dx}\left[\log_a x\right] = \frac{1}{\ln a} \cdot \frac{1}{x}$ We can put this together.

$\frac{d}{dx}\left[\log_a x\right] = \frac{1}{x\ln a}$ or $\frac{d}{dx}\left[\log_a u\right] = \frac{u'}{u\ln a}$

EXAMPLE: Find the derivative: $y = 8^{x^2}$.

Here we will apply the formula $\frac{d}{dx}\left[a^u\right] = (\ln a)a^u \cdot u'$. In this case, $u = x^2$, $u' = 2x$, and $a = 8$. So applying the formula you will get: $y' = (\ln 8) \cdot 8^{x^2} \cdot 2x$. Reordering will give us: $y' = 2x\ln 8\left(8^{x^2}\right)$ as the answer.

EXAMPLE: Find the derivative: $y = x\left(6^{-2x}\right)$.

We need to use the product rule on this one. Note: the derivative of 6^{-2x} is: $(\ln 6)6^{-2x}(-2)$:
$y' = x(\ln 6)6^{-2x}(-2) + 6^{-2x}(1)$. We will get: $y' = -2x(\ln 6)6^{-2x} + 6^{-2x}$. We can factor out a 6^{-2x} to get:
$y' = 6^{-2x}(1 - 2x\ln 6)$.

EXAMPLE: Find the derivative: $y = \log_4(2 + x\ln 4)$.

For this problem we will follow the formula $\frac{d}{dx}[\log_a u] = \frac{u'}{u\ln a}$. In this case, $u = 2 + x\ln 4$. Then $u' = \ln 4$.
This is because the derivative of 2 is zero, and the derivative of $u = x\ln 4$ is just *ln* 4 since this is considered a constant times *x*. Now we will put these into the formula: $y' = \frac{\ln 4}{(2 + x\ln 4)\cdot \ln 4}$. The final answer is

$$y' = \frac{1}{(2 + x\ln 4)}.$$

EXAMPLE: Find the derivative: $y = \log_3 \frac{x\sqrt{x-1}}{2}$. Write your answer as a single fraction.

This can be rewritten as: $y = \log_3 \frac{1}{2}x(x-1)^{\frac{1}{2}}$. We can use log property #2 to break this one up:

$h(x) = \log_3 \frac{1}{2} + \log_3 x + \log_3 (x-1)^{\frac{1}{2}}$. Then we can use log property #3: $y = \log_3 \frac{1}{2} + \log_3 x + \frac{1}{2}\log_3(x-1)$.
Now we want to take the derivative. The first term will drop out since it is a constant. For the third term, u is x – 1 and u' is 1: $y' = \frac{1}{x\ln 3} + \frac{1}{2}\cdot\frac{1}{(x-1)\ln 3}$. Now we want to get common denominators:

$y' = \frac{1}{x\ln 3}\cdot\frac{2(x-1)}{2(x-1)} + \frac{1}{2(x-1)\ln 3}\cdot\frac{x}{x}$. This will give us: $y' = \frac{2(x-1)+x}{2x(x-1)\ln 3}$. The top can be simplified:

$$y' = \frac{3x-2}{2x(x-1)\ln 3}.$$

EXAMPLE: Find the derivative: $y = \log\left(\frac{\sin x \sec x}{e^x \cdot 3^x}\right)^{\ln 10}$. Write your answer as a single fraction.

We can use log property #2 to break this up: $y = \ln 10\left(\log(\sin x \sec x) - \log\left(e^x 3^x\right)\right)$. Then apply log property #1:
$y = \ln 10\left(\log(\sin x) + \log(\sec x) - \log e^x - \log 3^x\right)$. Now we will take the derivative of each term separately by applying the formula $\frac{d}{dx}[\log_a u] = \frac{u'}{u\ln a}$. Since there is no base with this log, it is assumed to be base 10. For the term $\log(\sin x)$, $u = \sin x$, then $u' = \cos x$. For the term $\log(\sec x)$, $u = \sec x$, then $u' = \sec x \tan x$. For the term $\log\left(e^x\right)$, $u = e^x$, then $u' = e^x$. For the term $\log\left(3^x\right)$, $u = 3^x$, then $u' = 3^x \ln 3$.

Our derivative becomes: $y' = \ln 10\left(\dfrac{\cos x}{\sin x \ln 10} + \dfrac{\sec x \tan x}{\sec x \ln 10} - \dfrac{e^x}{e^x \ln 10} - \dfrac{3^x \ln 3}{3^x \cdot \ln 10}\right)$. The ln(10) will cancel out:

$y' = \dfrac{\cos x}{\sin x} + \dfrac{\sec x \tan x}{\sec x} - \dfrac{e^x}{e^x} - \dfrac{3^x \ln 3}{3^x}$. Now simplify: $y' = \cot x + \tan x - 1 - \ln 3$.

EXAMPLE: Find the derivative: $y = \theta \cdot \log_2\left(e^{(\cos\theta)(\ln 2)}\right)$.

First we can use log property #3: $y = \theta \cdot (\cos\theta)(\ln 2)\log_2 e$. We can rewrite this as: $y = (\ln 2)\log_2 e \cdot \theta \cdot \cos\theta$ since $(\ln 2)\log_2 e$ is a constant. Let's take a look at $(\ln 2)\log_2 e$ and see if we can simplify it. We can apply the change of base formula on the base 2 log: $(\ln 2)\log_2 e = \ln 2 \cdot \dfrac{\ln e}{\ln 2}$. Since $\ln e = 1$, then $(\ln 2)\log_2 e = 1$. So now our problem becomes $y = \theta\cos\theta$. We need to use a product rule on this one. In this case $f = \theta$ and $g = \cos\theta$. So $y' = \theta \cdot -\sin\theta + \cos\theta(1)$. We can simplify this: $y' = \cos\theta - \theta\sin\theta$.