

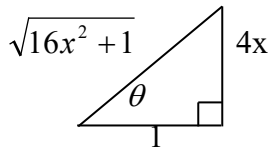
3.9 Derivatives of Inverse Trigonometric Functions

From trigonometry, we know that $\sin 30^\circ = \frac{1}{2}$. We put in an angle and get a value as a result. Recall that for inverse trig functions we put in the value and get an angle: $\sin^{-1} \frac{1}{2} = 30^\circ$. So here we put in the value of one half and got 30 degrees as a result. At the beginning of this course (Section 1.3) we did a brief review over inverse trigonometric functions. Now, let's use them for algebraic expressions.

EXAMPLE: Write in algebraic form: $\sec(\tan^{-1} 4x)$.

These problems involve drawing a triangle and labeling the sides with algebraic expressions. For all these problems we will assume that x is positive and the triangle should be drawn in the first quadrant. We can rewrite our problem as: $\sec\left(\tan^{-1} \frac{4x}{1}\right)$ We know that the adjacent side is 1 and the opposite side is $4x$. We can

use the Pythagorean theorem to find the hypotenuse: $c^2 = (4x)^2 + (1)^2$. So we have $c = \sqrt{16x^2 + 1}$

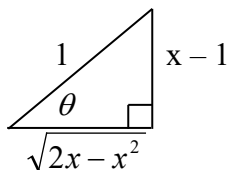


The secant on the outside of our problem tells us how to write our answer. From our drawing, secant is $\sqrt{16x^2 + 1}$ over 1 so we write our answer as:

$$\sec(\tan^{-1} 4x) = \sqrt{16x^2 + 1}$$

EXAMPLE: Write in algebraic form: $\cot(\sin^{-1}(x-1))$.

These problems involve drawing a triangle and labeling the sides with algebraic expressions. For all these problems we will assume that x is positive and the triangle should be drawn in the first quadrant. We can rewrite our problem as: $\cot\left(\sin^{-1} \frac{x-1}{1}\right)$ We know that the opposite side is $x-1$ and the hypotenuse is 1. We can use the Pythagorean theorem to find the hypotenuse: $1^2 = (x-1)^2 + a^2$. So we have $1 = x^2 - 2x + 1 + a^2$. Solving for a^2 we get $a^2 = 2x - x^2$, so $a = \sqrt{2x - x^2}$. Now we can draw the triangle.



The cotangent on the outside of our problem tells us how to write our answer. From our drawing, cotangent is $\sqrt{2x - x^2}$ over $x-1$ so we write our answer as:

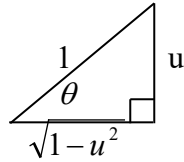
$$\cot(\sin^{-1}(x-1)) = \frac{\sqrt{2x - x^2}}{x-1} \text{ where } x \neq 1.$$

Derivative of inverse trig functions

Let's find the derivative of $\sin^{-1} u$. In order to do this we will draw a triangle like in the previous problems.

We can rewrite this problem as $\sin^{-1} \frac{u}{1}$. Then we know the hypotenuse is 1 and the opposite side is u .

Then we can use the Pythagorean Theorem to find the remaining side. You will get this triangle:



On our problem, let's let $y = \sin^{-1} u$. The definition of the inverse sine tells us that $\sin y = u$. Now let's take the derivative of both sides with respect to x using implicit differentiation:

$$\frac{d}{dx} [\sin y] = \frac{du}{dx}$$

You will need to use the chain rule on the left side.

$$\cos y \cdot y' = u'$$

Now solve for y'

$$y' = \frac{u'}{\cos y}$$

From our triangle we know that $\cos y = \frac{\sqrt{1-u^2}}{1}$. Plug this in for $\cos y$.

$$y' = \frac{u'}{\frac{\sqrt{1-u^2}}{1}}$$

Now simplify and we have our answer.

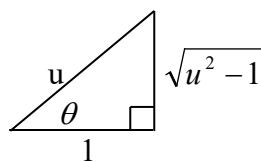
$$y' = \frac{u'}{\sqrt{1-u^2}}$$

So if $y = \sin^{-1} u$ then $y' = \frac{u'}{\sqrt{1-u^2}}$.

EXAMPLE: If $y = \sec^{-1} u$, find y' .

In order to do this we will draw a triangle like in the previous problems. We can rewrite this problem as $\sec^{-1} \frac{u}{1}$. Then we know the hypotenuse is u and the adjacent side is 1.

Then we can use the Pythagorean Theorem to find the remaining side. You will get this triangle:



The definition of the inverse secant tells us that $\sec y = u$. Now let's take the derivative of both sides with respect to x using implicit differentiation:

$$\frac{d}{dx}[\sec y] = \frac{du}{dx}$$

You will need to use the chain rule on the left side.

$$\sec y \tan y \cdot y' = u'$$

Now solve for y'

$$y' = \frac{u'}{\sec y \tan y}$$

We know that $\sec y = u$ and $\tan y = \sqrt{u^2 - 1}$ from our triangle. Now substitute.

$$y' = \frac{u'}{u\sqrt{u^2 - 1}}$$

So if $y = \sec^{-1} u$ then $y' = \frac{u'}{u\sqrt{u^2 - 1}}$.

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}[\sin^{-1} u] = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\cos^{-1} u] = -\frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{u'}{1+u^2}$$

$$\frac{d}{dx}[\cot^{-1} u] = -\frac{u'}{1+u^2}$$

$$\frac{d}{dx}[\sec^{-1} u] = \frac{u'}{|u|\sqrt{u^2 - 1}}$$

$$\frac{d}{dx}[\csc^{-1} u] = -\frac{u'}{|u|\sqrt{u^2 - 1}}$$

EXAMPLE: Find the derivative: $y = \sec^{-1}(2x)$.

We will let $u = 2x$. Then $u' = 2$. We just need to substitute these into the formula $\frac{u'}{|u|\sqrt{u^2 - 1}}$. You will get:

$$y' = \frac{2}{|2x|\sqrt{(2x)^2 - 1}}. \text{ Now we can simplify: } y' = \frac{1}{|x|\sqrt{4x^2 - 1}}. \text{ This is as far as we can go.}$$

EXAMPLE: Find the derivative: $y = \cos^{-1}(5x^4)$.

We will let $u = 5x^4$. Then $u' = 20x^3$. We just need to substitute these into the formula $-\frac{u'}{\sqrt{1-u^2}}$. You will

$$\text{get: } y' = \frac{-20x^3}{\sqrt{1-(5x^4)^2}}. \text{ Now we can simplify: } y' = \frac{-20x^3}{\sqrt{1-25x^8}}. \text{ This is as far as we can go.}$$

EXAMPLE: Find the derivative: $y = \cot^{-1}(3x^2 - 8)$.

We will let $u = 3x^2 - 8$. Then $u' = 6x$. We just need to substitute these into the formula $\frac{-u'}{1+u^2}$. You will get:

$$y' = \frac{-6x}{1+(3x^2-8)^2}. \text{ Now we can simplify: } y' = \frac{-6x}{1+9x^4-48x^2+64}. \text{ Our final answer is}$$

$$y' = \frac{-6x}{9x^4-48x^2+65}. \text{ This is as far as we can go.}$$

EXAMPLE: Find the derivative: $y = \tan^{-1}\sqrt{5-2x}$.

We will let $u = \sqrt{5-2x}$. Then $u' = \frac{1}{2}(5-x)^{-\frac{1}{2}}(-2)$. This simplifies to $u' = -\frac{1}{\sqrt{5-2x}}$. Now we need to

substitute these into the formula $\frac{u'}{1+u^2}$. You will get: $y' = -\frac{1}{1+(\sqrt{5-2x})^2}$. Simplifying gives us:

$$y' = -\frac{1}{1+5-2x} = -\frac{1}{6-2x} = -\frac{1}{\sqrt{5-2x}} \cdot \frac{1}{6-2x}. \text{ So our final answer is: } y' = -\frac{1}{(6-2x)\sqrt{5-2x}}.$$

EXAMPLE: Find the derivative: $y = \ln(\cot^{-1} 2x^3)$.

Since we have a natural log we will let $u = \cot^{-1} 4x^3$. Then $u' = -\frac{6x^2}{1+(2x^3)^2}$. This simplifies: $u' = -\frac{6x^2}{1+4x^6}$. To

take the derivative of the \ln we will use the formula $\frac{u'}{u}$. This will give us: $y = \frac{-\frac{6x^2}{1+4x^6}}{\cot^{-1} 4x^3}$. Now we can

simplify: $y' = -\frac{6x^2}{\cot^{-1} 2x^3 \cdot (1+4x^6)}$. Nothing more we can do on this one.

EXAMPLE: Find the derivative: $y = x \cdot \tan^{-1}(2x) - \frac{1}{4} \ln(1+4x^2)$.

We need to use the product rule on this one. When we take the derivative of $\tan^{-1}(2x)$ we will let $u = 2x$.

Then $u' = 2$ and we will use the formula the formula $\frac{u'}{1+u^2}$. For the second term since we have a natural log

we will let $u = 1+4x^2$. Then $u' = 8x$. To do a derivative of the \ln we will use the formula $\frac{u'}{u}$.

Using the product rule we get: $y' = x \cdot \frac{2}{1+(2x)^2} + \tan^{-1}(2x) \cdot (1) - \frac{1}{4} \cdot \frac{8x}{1+4x^2}$. Now we can simplify:

$$y' = x \cdot \frac{2}{1+4x^2} + \tan^{-1}(2x) - \frac{2x}{1+4x^2}. \text{ The first and last terms cancel and then we have: } y' = \tan^{-1}(2x).$$

EXAMPLE: Find the derivative: $y = 25 \sin^{-1}\left(\frac{x}{5}\right) - x\sqrt{25-x^2}$. Write your answer as a single fraction.

For the first term we will let $u = \frac{x}{5}$. Then $u' = \frac{1}{5}$. You will use $\frac{u'}{\sqrt{1-u^2}}$. For the second term you will need to use the product rule combined with the chain rule. Putting this all together you will have:

$$y' = 25 \cdot \frac{\frac{1}{5}}{\sqrt{1-\left(\frac{x}{5}\right)^2}} - x \cdot \frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x) - \sqrt{25-x^2}(1) \quad \text{Now we need to simplify.}$$

$$y' = \frac{5}{\sqrt{1-\frac{x^2}{25}}} + \frac{x^2}{\sqrt{25-x^2}} + \sqrt{25-x^2} \quad \text{For } 1-\frac{x^2}{25} \text{ we can get common denominators: } \frac{25-x^2}{25}.$$

$$y' = \frac{5}{\sqrt{\frac{25-x^2}{25}}} + \frac{x^2}{\sqrt{25-x^2}} + \sqrt{25-x^2} \quad \text{This simplified further.}$$

$$y' = \frac{5}{\frac{\sqrt{25-x^2}}{5}} + \frac{x^2}{\sqrt{25-x^2}} + \sqrt{25-x^2} \quad \text{The first term can be simplified further.}$$

$$y' = \frac{25}{\sqrt{25-x^2}} + \frac{x^2}{\sqrt{25-x^2}} + \frac{\sqrt{25-x^2}}{1} \quad \text{Multiply the last term by } \frac{\sqrt{25-x^2}}{\sqrt{25-x^2}} \text{ to get common denominators.}$$

$$y' = \frac{25+x^2-(25-x^2)}{\sqrt{25-x^2}}. \quad \text{Now simplify the numerator and we are done.}$$

$$y' = \frac{2x^2}{\sqrt{25-x^2}}.$$