4.1 Maxima and Minima

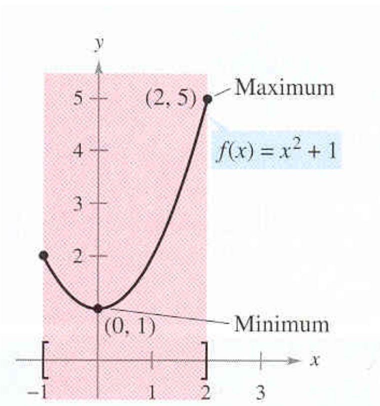
One application of the derivative is finding minimum and maximum values off a graph. In precalculus we were only able to do this with quadratics by find the vertex. Derivatives will allow us to find these values for functions with higher powers. The word extrema refers to min and max values.

Extrema of a Function

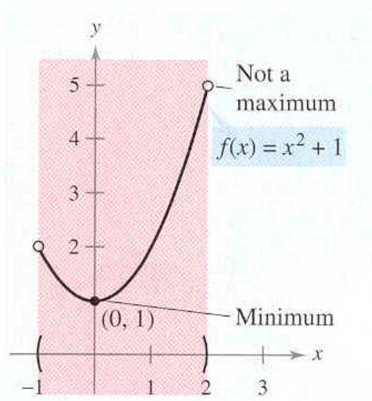
1.) $f(c)$ is the minimum of *f* on interval I if $f(c) \le f(x)$ for all x in I.

2.) $f(c)$ is the maximum of *f* on interval I if $f(c) \ge f(x)$ for all x in I.

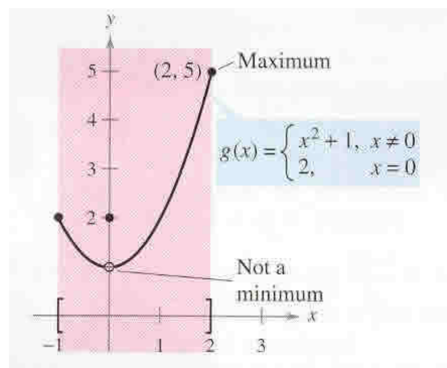
A function is guaranteed to have a minimum and maximum value if the function is continuous on a closed interval. The following are examples of when a function may or may not have a min or max:



(a) f is continuous, $[-1, 2]$ is closed.



(b) f is continuous, $(-1, 2)$ is open.

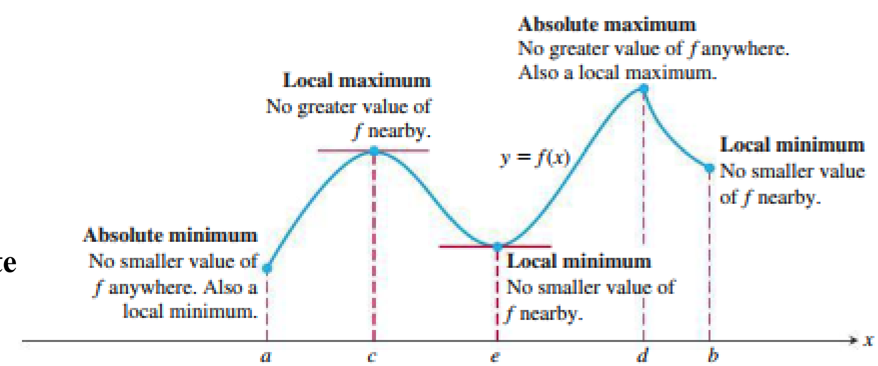


(c) g is not continuous, $[-1, 2]$ is closed. Extrema can occur at interior points or endpoints of an interval. Extrema that occur at the endpoints are called **endpoint extrema.**

From the pictures above we see that in picture (a) we have a closed interval and there is a defined maximum and minimum value. The highest point in and interval is call the absolute maximum and the lowest point in the interval is called a absolute minimum. Picture (b) has an absolute minimum, but not a maximum since that point is not included in our interval. Picture (c) has a absolute maximum but not a minimum on [-1, 2] because this graph is not continuous on this interval.

Local Extreme Values

Local extreme values means there is a hill or valley in the graph. This is not necessarily the absolute maximum or minimum of the graph. In the picture to the right we see a hill and valley but they are not the highest and lowest point of the graph. The highest or lowest possible points on the graph are called **absolute extrema**, and these are labeled on the graph.



Critical Number

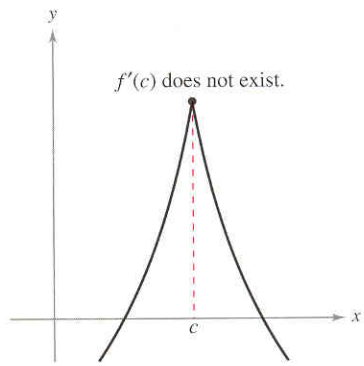
Let *f* be defined at c. The point x = c is a critical number if **either** of the following occurs which is shown in the pictures below:

1.) $f'(c)$ is undefined <u>OR</u>

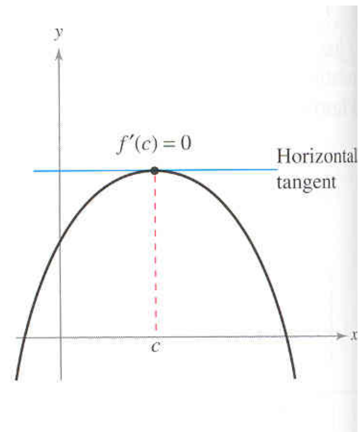
2.) $f'(c) = 0$

<u>AND</u>

$f(c)$ is defined.



c is a critical number of f.



The derivative would not exist for any graphs that have a corner. The absolute value graph would apply here because this has a sharp corner. Here's why:

EXAMPLE: Find the derivative of $y = |x|$ at the point (0, 0) by using the limit process.

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ Start with the limit formula.

$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ In this case x = 0.

$\lim_{h \to 0} \frac{|h| - 0}{h}$ We put h in for x and we know that $f(0) = 0$.

$\lim_{h \to 0} \frac{|h|}{h}$ As x approaches 0 from the left, this expression will equal -1. As x approaches 0 from the right, this express will equal 1. Since this approaches two different values, the limit does not exist and therefore the derivative does not exist at (0, 0).

How to find extrema on a closed interval [a, b]

1.) Take the first derivative and set it equal to zero to find the critical point. You may also need to see if the derivative is undefined anywhere in the interval, because this will also give you a critical point.

2.) Evaluate *f* at each critical point you find in the interval.

3.) Evaluate *f* at the endpoints of your interval. (In other words, find $f(a)$ and $f(b)$).

4.) The least of these values is the absolute minimum. The greatest is the absolute maximum.

EXAMPLE: Let $f(x)=3x^4+4x^3-12x^2$ on [-4, 2]. Find all critical numbers and absolute extrema on this interval.

The first thing to do is to find the derivative. We will get $f'(x)=12x^3+12x^2-24x$. We need to set this to zero to find the critical point: $12x^3+12x^2-24x=0$. We need to factor this and set the factors equal to zero. After factoring you should get $12x(x^2+x-2)$ = $12x(x-1)(x+2)$. Setting this equal to zero we will get the critical points 0, 1, and -2. There are no places where the derivative will be undefined, so these are the only critical points. Now we need to set up a table with our critical points and also the endpoints. We will use the x values 0, 1, and -2 since they are the critical points. We also need to check x = -4 and x = 2 since these are the endpoints of our interval. So we will put all these x-values into the ORIGINAL function to get the y-values. DO NOT use the derivative. For example, to get 320 on the table below I put -4 into the original equation for x: $f(-4)=3(-4)^4+4(-4)^3-12(-4)^2=320$.

x	-4	-2	0	1	2
$f(x)$	320	-32	0	-5	32

After you complete the table we need to find the absolute extrema. The largest y-value will be the absolute maximum and the smallest y-value will be the absolute minimum. You would write the following as the answer: The absolute maximum is 320 when x = -4. The absolute minimum is -32 when x = -2.

EXAMPLE: Let $f(x)=12\cdot\sqrt[3]{x^2}-8x$ on [-1, 2]. Find all critical numbers and absolute extrema on this interval.

We can rewrite this as; $f(x)=12x^{\frac{2}{3}}-8x$. When we find the derivative we can just use the power rule. You will get: $f'(x)=8x^{-\frac{1}{3}}-8$, which can be rewritten as $f'(x)=\dfrac{8}{\sqrt[3]{x}}-8$. By observation we can see that a zero would cause the derivative to be undefined, so we know that x = 0 is a critical point since this is defined in the original function. We need to set this equal to zero to find the other critical point: $0=\dfrac{8}{\sqrt[3]{x}}-8$. We can add the 8 to both sides to get $8=\dfrac{8}{\sqrt[3]{x}}$. Cross multiplying we will get $8\cdot\sqrt[3]{x}=8$. Then divide both sides by 8 to get $\sqrt[3]{x}=1$. After cubing both sides we get x = 1 which is our second critical point. When we make our table we will be using the x = -1, 0, 1, and 2. To get the y-values, we will put these in for x in $f(x)=12\cdot\sqrt[3]{x^2}-8x$.

x	-1	0	1	2
$f(x)$	20	0	4	3.05

We see that the absolute maximum is 20 when x = -1 and the absolute minimum is 0 when x = 0.

EXAMPLE: Let $f(\theta) = \sec\theta$ on $\left[-\frac{\pi}{6}, \frac{\pi}{3}\right]$. Find all critical numbers and absolute extrema on this interval.

First the derivative is $f'(\theta) = \sec\theta\tan\theta$. By looking at this we find that $\theta = \frac{\pi}{2}$ would make the derivative undefined. If this were in our interval then this would be a critical point. Since it is not in our interval we will ignore this and just set the derivative equal to zero: $0 = \sec\theta\tan\theta$. When we do this we have $\sec\theta = 0$ and $\tan\theta = 0$. If we try to set the first equation equal to zero we can write $\sec\theta = \frac{1}{\cos\theta}$. so our equation becomes $0 = \frac{1}{\cos\theta}$. Cross multiplying we will get 0 = 1, which is not true so this first equation won't give us any critical points. The second equation, $\tan\theta = 0$ can be rewritten as $\frac{\sin\theta}{\cos\theta} = 0$. Cross multiplying we will get $\sin\theta = 0$ which we can solve to get $\theta = 0$. This will be the only critical point. So we will make our table with our endpoints and also $\theta = 0$. When we put in $\theta = -\frac{\pi}{6}$ we will get $\frac{2\sqrt{3}}{3} = 1.15$. It is okay to use the decimal in the table so we can easily compare the results.

θ	$-\frac{\pi}{6}$	0	$\frac{\pi}{3}$
$f(\theta)$	1.15	1	2

We see that the absolute maximum is 2 when $x = \frac{\pi}{3}$. The absolute minimum is 1 when x = 0.

EXAMPLE: Let $f(x) = e^{-x^2}$ on [-2, 1]. Find all critical numbers. Then find the absolute extrema on this interval.

First the derivative is $f'(x) = e^{-x^2} \cdot -2x$. So $f'(x) = -2xe^{-x^2}$. Now we set the derivative equal to zero: $0 = -2xe^{-x^2}$. When we do this we have $-2x = 0$ and $e^{-x^2} = 0$. So one critical number is $x = 0$. For the second equation, the only way to solve this would be to take the natural log of both sides to clear out the e. However when we do this we cannot take the natural log of zero. Therefore this equation has no solution. Therefore the only critical number is $x = 0$. Now we need to set up a table with our critical points and also the endpoints. We will use the x values -2, 0, and 1. So we will put all these x-values into the ORIGINAL function to get the y-values.

x	-2	0	1
$f(x)$	e^{-4}	1	e^{-1}

So we see the absolute maximum will be 1 and that occurs at $x = 0$. Then the absolute minimum is e^{-4} and that occurs at $x = -2$.

EXAMPLE: Let $f(x)=\frac{2x+5}{3}$ on [-1, 5]. Find all critical numbers. Then find the absolute extrema on this interval.

We can rewrite this as $f(x)=\frac{2}{3}x+\frac{5}{3}$ by dividing both things on top by 3. Now take the derivative and you will get: $f'(x)=\frac{2}{3}$. There are no places where the derivative will be undefined and we can't set the derivative equal to zero because there is no variable. Because of this we know that there are no critical points. If there are no critical points then the only numbers we will use in our table will be the endpoints, x = -1 and x = 5.

x	-1	5
f(*x*)	1	5

EXAMPLE: Find the extreme values (absolute and local) of the function over its natural domain, and where they occur: $f(x)=\frac{2}{x^2-1}$.

We can rewrite the above as $f(x)=2(x^2-1)^{-1}$. Now we want to take the derivative. You will use the chain rule. You will get $f'(x)=-2(x^2-1)^{-2}(2x)$. This can be rewritten as $f'(x)=\frac{-4x}{(x^2-1)^2}$. We will set this equal to zero to get a critical point. $0=\frac{-4x}{(x^2-1)^2}$. After cross multiplying you will get $0=-4x$, so $x=0$. If there is a point that causes the derivative function to be undefined, then this is another critical point. In this problem x = 1 and x = -1 would cause the bottom to be zero. We cannot have any extrema at these points since they are not defined on the original function. Therefore, the only critical number is x = 0. We need to decide if this is a local minimum or maximum. Let's make a table of values close to 0:

x	-0.5	0	0.5
f(*x*)	-2.667	-2	-2.667

We see that as we approach zero from the left and the right the y values are getting bigger. This means there is a local maximum of –2 at x = 0.

Does this have an absolute maximum or absolute minimum? The answer is no. Since we have vertical asymptotes at x = 1 and x = –1 this means as we very close to 1 or –1 the y values will approach positive or negative infinity. This is not an exact number therefore there is no absolute maximum or minimum. So the only extrema is a local maximum of –2 at x = 0.

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EXAMPLE: Find the extreme values (absolute and local) of the function over its natural domain, and where they occur: $f(x) = \cos^{-1}(e^x)$.

We will take the derivative. We will use the formula $f'(x) = \dfrac{-u'}{\sqrt{1-u^2}}$ where $u = e^x$. You will get:

$f'(x) = \dfrac{-e^x}{\sqrt{1-(e^x)^2}}$. This can be written as: $f'(x) = \dfrac{-e^x}{\sqrt{1-e^{2x}}}$. Now we need to find the critical points. The first thing we will do is to set the derivative equal to zero: $0 = \dfrac{-e^x}{\sqrt{1-e^{2x}}}$. We cross multiply to get $0 = -e^x$. To solve we need to take the natural log of both sides, however we cannot take the natural log of zero. Therefore we did not find any critical numbers this way. The other way to find critical numbers is to see what number makes the derivative undefined but still defined in the original function. The derivative is a rational expression which means the way it can be undefined is if the denominator is zero. So we will set the denominator equal to zero: $0 = \sqrt{1-e^{2x}}$. After squaring both sides: $0 = 1 - e^{2x}$. Then we can isolate the e: $e^{2x} = 1$. Take the natural log of both sides: $\ln e^{2x} = \ln 1$. This simplifies to $2x = 0$, so $x = 0$. This is defined in the original function so this is a critical number. We now need to decide if this is a local minimum or maximum. Let's make a table of values close to 0:

x	-100	-10	-2	-1	0
$f(x)$	1.5708	1.5708	1.43	1.19	0

We see here that since the numbers are getting constant the farther we go to the left, this means there is a horizontal asymptote. Specifically, this is $y = \dfrac{\pi}{2}$. This means there will not be an absolute max because the graph will get very close to but never touch $y = \dfrac{\pi}{2}$. So we only have an absolute minimum. Therefore the absolute minimum is 0 at $x = 0$.