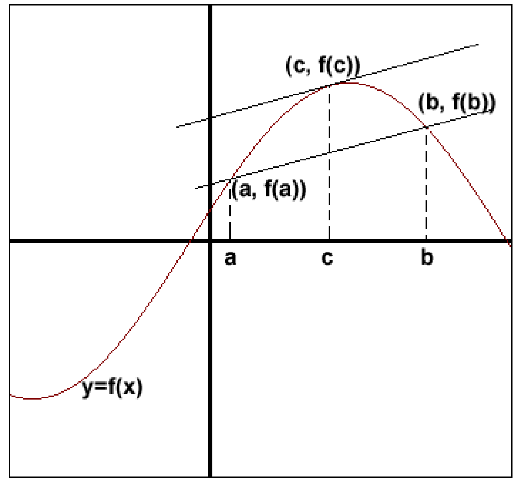
4.2 The Mean Value Theorem

Looking at the picture to the right I can find two points such that the slope of the line going through these two points is the same as the slope of a line going through point x. This is called the



Mean Value Theorem.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In order for the Mean Value Theorem to be applied, f must be continuous on [a, b] and differentiable on (a, b). Then c must be on (a, b).

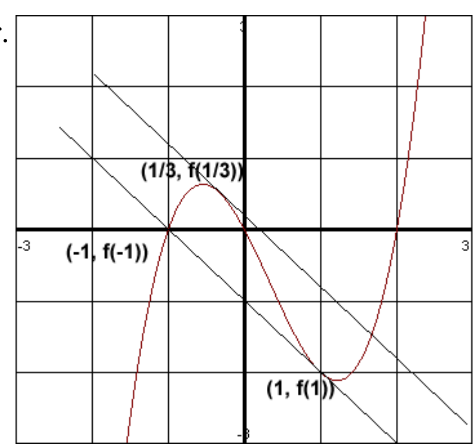
EXAMPLE: Determine whether the Mean Value Theorem can be applied to $f(x) = x^{\frac{4}{5}}$ on [0, 1].

This function is continuous on [0, 1]. The derivative of this is $f'(x) = \frac{4}{5x^{\frac{1}{5}}}$. It is not defined at zero. This doesn't matter because it has to be differentiable on (0, 1) which it is since zero is not included.

EXAMPLE: Determine whether the Mean Value Theorem can be applied to $f(x) = x(x^2 - x - 2)$ on [-1, 1]. If yes, then find all values of *c* on (*a*, *b*) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

We know f is continuous on [-1, 1]. To find the derivative, we rewrite f as $f(x) = x^3 - x^2 - 2x$. Then the derivative is $f'(c) = 3c^2 - 2c - 2$. Now let's find the other side of the equation, $\frac{f(b) - f(a)}{b - a}$. Plugging in -1 for a and 1 for b we get: $\frac{f(1) - f(-1)}{1 - (-1)} = \frac{-2 - 0}{2} = -1$. So now we set $f'(c) = -1$. We will get: $-1 = 3c^2 - 2c - 2$. Setting it equal to zero we get $0 = 3c^2 - 2c - 1$. Factoring will give us $0 = (3c + 1)(c - 1)$. Solving for c we get $c = -\frac{1}{3}, 1$. Both of these would be our answer.

In the graph to the right you can see what we just did. The line that connects the points at *x* = -1 and *x* = 1 has the same slope as at the point $x = \frac{1}{3}$.



EXAMPLE: Determine whether the Mean Value Theorem can be applied to $f(x) = \ln(x-1)$ on [2, 4]. If yes, then find all values of *c* on (*a*, *b*) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

The derivative is $f'(c) = \frac{1}{c-1}$. We know that f is differentiable on [2, 4]. Now let's look at our Mean Value Theorem formula. Now let's look at the right side of the formula, which is $\frac{f(b)-f(a)}{b-a}$. This is $\frac{f(4)-f(2)}{4-2} = \frac{\ln 3 - \ln 1}{2} = \frac{\ln 3 - 0}{2} = \frac{\ln 3}{2}$. So now $f'(c) = \frac{\ln 3}{2}$, or $\frac{1}{c-1} = \frac{\ln 3}{2}$. We need to solve this for c by cross multiplying: $\ln 3 \cdot (c-1) = 2$. Then $c - 1 = \frac{2}{\ln 3}$. Solving for c we get $c = 1 + \frac{2}{\ln 3} \approx 2.8$. This is in our interval, so it satisfies the Mean Value Theorem.

EXAMPLE: Determine whether the Mean Value Theorem can be applied to $f(x) = 2\sin x + \sin 2x$ on $[0,\ \pi]$. If yes, then find all values of *c* on (*a*, *b*) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

We know that f is continuous on our interval. The next thing to do is find the derivative:
$f'(c) = 2\cos(c) + 2\cos(2c)$ which is defined for all x on $(0,\ \pi)$ and therefore differentiable. Now let's look at the right side of the formula, which is $\frac{f(b)-f(a)}{b-a}$, which is $\frac{f(\pi)-f(0)}{\pi - 0} = \frac{0-0}{\pi} = 0$. So now $f'(c) = 0$. We will now solve $2\cos(c) + 2\cos(2c) = 0$. I will using the identity $\cos(2c) = 2\cos^2 c - 1$ which is a double angle formula. You will get: $2\cos(c) + 2(2\cos^2 c - 1) = 0$. After distributing we will get:
$2\cos(c) + 4\cos^2 c - 2 = 0$, or $4\cos^2(c) + 2\cos(c) - 2 = 0$. We can divide both sides by 2:
$2\cos^2(c) + \cos(c) - 1 = 0$. After factoring we get: $(\cos(c)+1)(2\cos(c)-1) = 0$. Solving this we get $\cos(c) = -1$, in which $c = \pi$. The other equation is $\cos(c) = \frac{1}{2}$. Solving this you get $c = \frac{\pi}{3}$. Both of these are in our interval, so it satisfies the Mean Value Theorem.

More on next page…

EXAMPLE: Determine whether the Mean Value Theorem can be applied to $f(x)=\sqrt{x(1-x)}$ on [0, 1]. If yes, then find all values of c on (a, b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Since we are only using numbers between 0 and 1 we know this is continuous. It is okay to take the square root of zero. To see if it is differentiable, we need to take the derivative. It would be easier to first distribute the x inside the square root so the product rule is not necessary. I will also change the square root into an exponent: $f(x)=\left(x-x^2\right)^{\frac{1}{2}}$. Now we take the derivative using the chain rule: $f(x)=\frac{1}{2}\left(x-x^2\right)^{-\frac{1}{2}}(1-2x)$.

This can be rewritten as: $f(x)=\dfrac{1-2x}{2\sqrt{x(1-x)}}$. We see that the derivative will be undefined at x = 0 and x = 1, however the Mean Value Theorem works on the open interval, which is (0, 1). Therefore 0 and 1 are not concluded so we know we can apply the Mean Value Theorem. So now we need to use the formula:

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\frac{1-2c}{2\sqrt{c(1-c)}}=\frac{f(1)-f(0)}{1-0}$$

We see that $f(1)=0$ and $f(0)=0$. So we can simplify.

$$\frac{1-2c}{2\sqrt{c(1-c)}}=\frac{0-0}{1}$$

Now we cross-multiply.

$$0=1-2c$$

Solve for c.

$$c=\frac{1}{2}$$

This number is in our interval, so it satisfies the Mean Value Theorem.