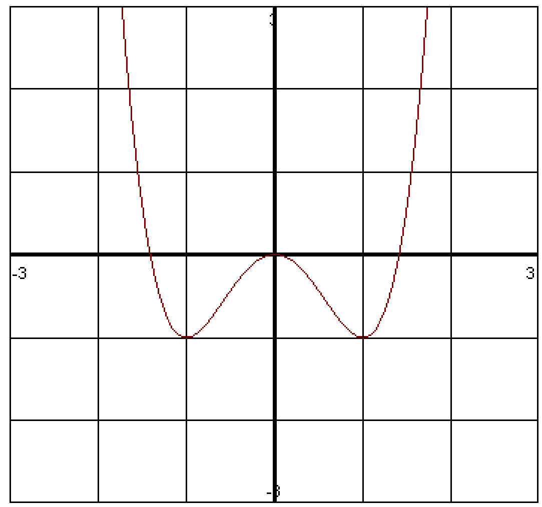
4.3 The First Derivative Test

In this section we will use the derivative to tell us if the graph is increasing, deceasing, or constant. If a function is **monotonic** on an interval, then the function is increasing or decreasing on that interval.

Increasing: as x increases, y increases
Decreasing: as x increases, y decreases
Constant: as x increases, y does not change.

EXAMPLE: Use the graph of $f(x) = x^4 - 2x^2$ to determine the interval(s) of increasing, decreasing, or constant. Indicate all extrema (max and min)

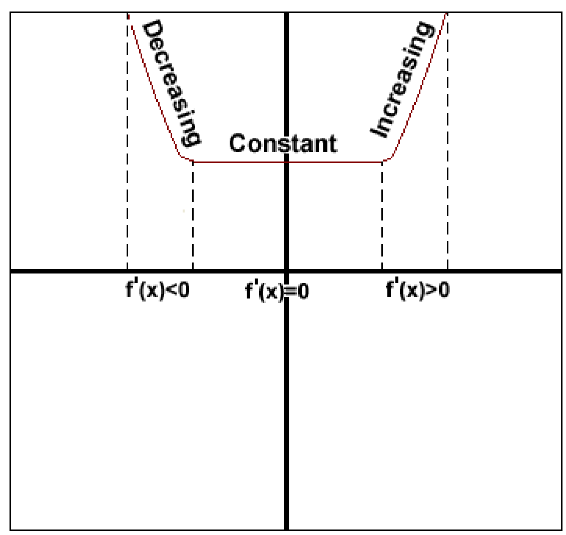


When we look for the interval(s) of increasing, as we move from left to right if the graph is going uphill then it is increasing. Likewise if we move from left to right and the graph is going downhill, then it is increasing.

We want to indicate the intervals that the graph is increasing. This will be $(-1,\ 0) \cup (1,\ \infty)$. We are indicating the X values in which this is increasing. For the decreasing intervals, it is $(-\infty,\ -1) \cup (0,\ 1)$.

The relative maximum is at (0, 0) and the relative minimum is at (-1, -1) and (1, -1). There is no absolute maximum, however the absolute minimum is -1.

Here we were able to read our values off of the graph. Suppose a graph is not given. Do we need to always graph each function in order to find the interval(s) of increasing and decreasing? The answer is no. We can use what is called the First Derivative Test.



Corollary 3

If $f'(x) > 0$ for all x in (a, b) then f is increasing on [a, b].

If $f'(x) < 0$ for all x in (a, b) then f is decreasing on [a, b].

If $f'(x) = 0$ for all x in (a, b) then f is constant on [a, b].

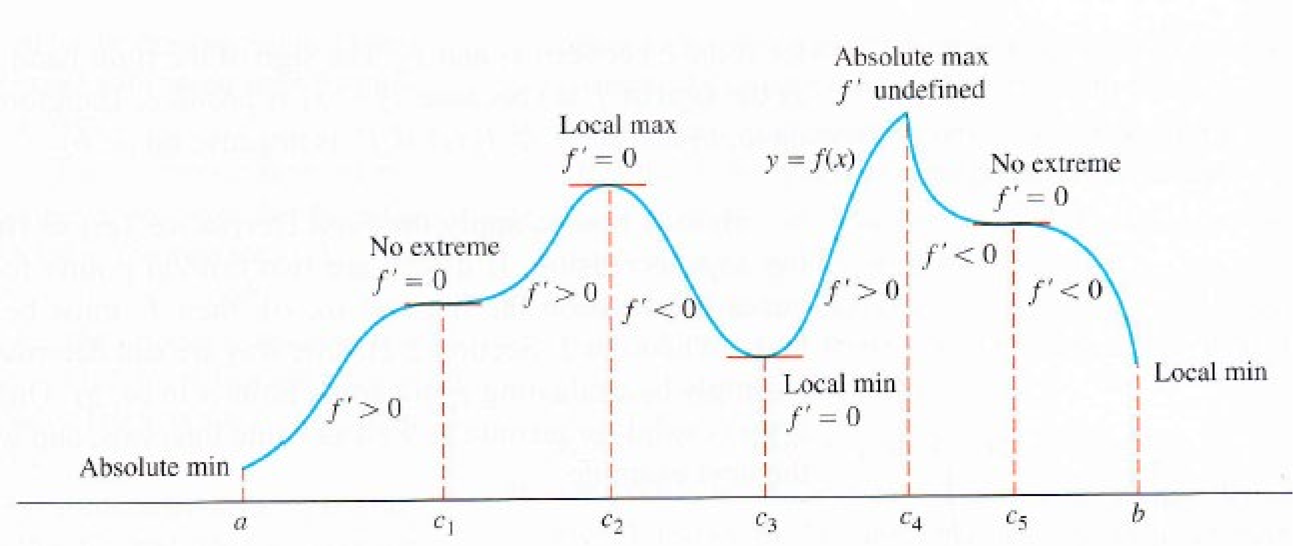
How to find interval(s) of increasing, decreasing, constant with no graph

1.) Find the critical numbers in (a, b) to determine test intervals

2.) Determine the sign of $f'(x)$ at one test value in each of the intervals

3.) Use Corollary 3 to determine if f is increasing or decreasing.

As mentioned, a function's first derivative tells how the graph rises and falls, as illustrated below:



Let's look at our graph example we started this section with.

EXAMPLE: Find the critical points of: $f(x) = x^4 - 2x^2$. Identify the intervals on which *f* is increasing or decreasing. Find the function's local maximum and minimum.

Following our instructions we need to first find the derivative: $f'(x) = 4x^3 - 4x$. Now we need to find the critical points by setting this equal to zero: $0 = 4x^3 - 4x$. Factoring this we get $0 = 4x(x^2 - 1)$. Solving for zero we get $x = 0,\ \ x = \pm 1$. Now we need to make a table with our critical values:

-1 0 1

Now we need to pick a test value that is in each of the regions above. In the first column I need to test number less than -1, so I will choose -2. In the second column I will test -0.5, since this is between -1 and 0. In the third column I will test 0.5 because it is in between 0 and 1. Finally I will test 2 since this is greater than 1. **When you test these values you need to put them into the DERIVATIVE function because we are looking at slopes.** If the result is a positive number we will use a + and if it is negative we will put in a –. After you do that your table should look like the following:

–	+	–	+

-1 0 1

Now we will use Corollary 3. If there is a negative then that means the graph has a negative slope, so it is decreasing. If there is a positive the graph has a positive slope and is increasing. We just need to write our answer in interval notation. You will get $(-1,\ 0) \cup (1,\ \infty)$ for the increasing intervals and $(-\infty,\ -1) \cup (0,\ 1)$ for the decreasing intervals.

So now how do we find the local maximum and minimum? We need to use the first derivative test.

First Derivative Test for Local Extrema

If $f'(x)$ changes from – to + at c then f has a local minimum at x = c.

If $f'(x)$ changes from + to – at c then f has a local maximum at x = c.

If $f'(x)$ does not change on both sides of c then c is neither a min or a max.

Looking at our above example we just did we see that at -1 and at 1 we have a – and then a +. This means there is a relative minimum at $x = \pm 1$. At zero we have + and – so there this means there is a relative max at x = 0. We need to indicate the point on the graph where the min and max is. For example, when we find the relative max, we will put a zero inot the ORIGINAL equation $f(x) = x^4 - 2x^2$. You will get (0, 0). For the relative minimum, we will put -1 and 1 into $f(x) = x^4 - 2x^2$. You will get the points (1, -1) and (-1, -1), which correspond to the ones on the graph.

EXAMPLE: Find the critical points of: $f(t) = 27t - t^3$. Identify the intervals on which *f* is increasing or decreasing. Find the function's local and absolute extreme values.

We need to find the derivative: $f'(t) = 27 - 3t^2$. Now we set this equal to zero: $0 = 27 - 3t^2$. After factoring we will get $0 = 3(9 - t^2) = 3(3 - t)(3 + t)$. Solving this we will get $t = \pm 3$. So this is where the critical points are at. Now we make our table:

-3 3

Now we need to pick a test value less than -3, so I will choose -4. Then I need to pick a test value between -3 and 3, so I will choose 0. Finally I will choose a test value greater than 3, so I will use 4. Once again, **you will put these values into the DERIVATIVE function**. After we do that we get the table below:

–	+	–

-3 3

We see that there is a relative minimum at t = -3 since we have a – and +. The corresponding point on the graph will be (-3, -54). There is a relative maximum at x = 3 since there is a + and -. The corresponding point on the graph will be (3, 54). As a reminder, to find the y values you need to put the critical points into the ORIGINAL equation, which is $f(t) = 27t - t^3$.

The interval of increasing is (-3, 3) since this is the region that has a positive. The intervals of decreasing are the ones with a negative sign, so we write: $(-\infty, -3) \cup (3, \infty)$. Notice that I am always using parenthesis to indicate regions of increasing and decreasing. This is because we don't want to include the critical points because here the derivative is zero so it is not increasing or decreasing at these points. So which of these are absolute? The original graph is a cube, so it has at most two turning points. Since there is a negative in front of the t with the highest power, the graph will rise to the left and fall to the right, which means that there is no absolute max or min.

EXAMPLE: Find the critical points of: $f(x) = (x-2)e^{-3x}$. Identify the intervals on which *f* is increasing or decreasing. Find the function's local maximum and minimum.

We need to find the derivative, and this will involve the product rule: $f'(x) = (x-2)\cdot -3e^{-3x} + e^{-3x}(1)$. This simplifies to: $f'(x) = -3e^{-3x}(x-2) + e^{-3x}$. Now we set this equal to zero: $0 = -3e^{-3x}(x-2) + e^{-3x}$. To solve we need to factor: $0 = e^{-3x}[-3(x-2)+1]$. This simplifies: $0 = e^{-3x}[7-3x]$. Now set each piece equal to zero. To solve $0 = e^{-3x}$ you want to take the natural log of both sides, however *ln*(0) is not defined. Therefore this will not result in any solutions. So now solve $7 - 3x = 0$ to get $x = \frac{7}{3}$. So we have a critical point at $x = \frac{7}{3}$.

Now we make our table:

7/3

Now we need to pick a test value less than 7/3, so I will choose 0. Then I need to pick a test value greater than 7/3, so I will use 3. Once again, **you will put these values into the DERIVATIVE function**. After we do that we get the table below:

+	–

7/3

We see that there is a local maximum at $x = 7/3$ since we have a + and –. The corresponding point on the graph will be $\left(\frac{7}{3}, e^{-\frac{1}{3e^{7}}}\right)$. As a reminder, to find the *y* values of the local maximum (or minimum) you need to put the critical points into the ORIGINAL equation, which is $f(x) = (x-2)e^{-3x}$. The interval of increasing is $\left(-\infty, \frac{7}{3}\right)$ since this is where we see a plus sign on our table and the interval of decreasing is $\left(\frac{7}{3}, \infty\right)$ since this is where we see a negative.

EXAMPLE: Find the critical points of: $f(x) = \frac{\sqrt{9-x^2}}{x^2}$. Identify the intervals on which *f* is increasing or decreasing. Find the function's local and absolute extreme values.

Before we get started it is good to know the function's domain so that we know what type of answers are acceptable. We know that zero is not in the domain since it makes the denominator zero. Then we look inside the radical. We want to solve this inequality: $9 - x^2 \geq 0$. This gives us the interval: $[-3,\ 0) \cup (0,\ 3]$. Now we need to find the derivative. I could use the quotient rule here, but instead I will rewrite this as $f(x) = x^{-2}\sqrt{9-x^2}$ and apply the product rule . By the product rule:

$f'(x) = x^{-2}\frac{1}{2}(9-x^2)^{-\frac{1}{2}}(-2x) + \sqrt{9-x^2}(-2x^{-3})$ which can be rewritten as $f'(x) = \frac{-1}{x\sqrt{9-x^2}} - \frac{2\sqrt{9-x^2}}{x^3}$.

Now set it equal to zero to find any critical points:

$$0 = \frac{-1}{x\sqrt{9-x^2}} - \frac{2\sqrt{9-x^2}}{x^3}$$

$$\frac{1}{x\sqrt{9-x^2}} = -\frac{2\sqrt{9-x^2}}{x^3}$$ Cross multiply.

$$-2x(9-x^2) = x^3$$ Simplify and set equal to zero.

$$x^3 - 18x = 0$$ Factor.

$$x(x^2-18) = 0$$

Setting both equal to zero will give $x = 0$, and $x = \pm 3\sqrt{2}$. All of these are outside of the domain. The only critical point will be at zero since this makes the derivative undefined. Below is the sign chart for the derivative with the critical point. Notice I did not test a point less than -3 or greater than 3 since this falls outside the domain for this function. The sign chart would tell us that there is a local maximum at $x = 0$, however since zero is undefined on the original function then there is no relative extrema.

+	–

–3 0 3

The chart tells us the function is increasing on (–3, 0) and decreasing on (0, 3).

What about the absolute max? Since there is a vertical asymptote at $x = 0$ then there is no absolute maximum. The absolute minimums occur at the endpoints of the domain: (–3, 0) and (3, 0).

EXAMPLE: Find all extrema and interval(s) of increasing and decreasing for $f(\theta) = \sin\theta\cos\theta$ on $[0, \ \pi]$.

For the derivative we need to use the product rule: $f'(\theta) = \sin\theta(-\sin\theta) + \cos\theta(\cos\theta)$. This simplifies to: $f'(\theta) = \cos^2\theta - \sin^2\theta$. We need to set this equal to zero: $0 = \cos^2\theta - \sin^2\theta$. To solve this, we need to change all of these into sines or cosines. It doesn't matter which. I will use the identity: $\sin^2\theta = 1 - \cos^2\theta$. We will now have: $0 = \cos^2\theta - (1 - \cos^2\theta)$. Simplifying gives us $0 = 2\cos^2\theta - 1$. We need to solve this for x. First isolate the cosine: $\frac{1}{2} = \cos^2\theta$. We need to take the square root of both sides: $\pm\frac{1}{\sqrt{2}} = \cos\theta$. After rationalizing we get $\pm\frac{\sqrt{2}}{2} = \cos\theta$. We need to look at the unit circle to find what angles between 0 and π give us a value of $\pm\frac{\sqrt{2}}{2}$. The answer is: $\frac{\pi}{4}$, $\frac{3\pi}{4}$. On our table we need to include the endpoints of our interval.

0 $\frac{\pi}{4}$ $\frac{3\pi}{4}$ π

For the test points I will use 0, $\frac{\pi}{2}$, and π. Put these into the **DERIVATIVE** to get:

+	–	+

$0 \quad \frac{\pi}{4} \quad \frac{3\pi}{4} \quad \pi$

We have a relative maximum at $\frac{\pi}{4}$. The corresponding point is $\left(\frac{\pi}{4},\ \frac{1}{2}\right)$. We have a relative minimum at $\frac{3\pi}{4}$. The corresponding point is $\left(\frac{3\pi}{4},\ -\frac{1}{2}\right)$. The intervals of increasing are $\left(0,\ \frac{\pi}{4}\right)\cup\left(\frac{3\pi}{4},\ \pi\right)$. The intervals of decreasing are $\left(\frac{\pi}{4},\ \frac{3\pi}{4}\right)$.

EXAMPLE: Find the critical points of: $f(x) = 3x^{\frac{2}{3}}(x-4)$. Identify the intervals on which *f* is increasing or decreasing. Find the function's local and absolute extreme values.

We could use the product rule here, but instead I will first distribute: $f(x) = 3x^{\frac{5}{3}} - 12x^{\frac{2}{3}}$. Now we can apply the power rule: $f'(x) = 3\cdot\frac{5}{3}x^{\frac{2}{3}} - 12\cdot\frac{2}{3}x^{-\frac{1}{3}}$ which simplifies to: $f'(x) = 5x^{\frac{2}{3}} - \frac{8}{x^{\frac{1}{3}}}$. We see that $x = 0$ will make the derivative undefined, so this is one of our critical points since $x = 0$ is defined in the ORIGINAL function. Now we need to set the derivative equal to zero to see if there are any more critical points: $0 = 5x^{\frac{2}{3}} - \frac{8}{x^{\frac{1}{3}}}$. First we move one of the terms: $5x^{\frac{2}{3}} = \frac{8}{x^{\frac{1}{3}}}$ and then we can cross multiply: $5x = 8$ so then $x = \frac{8}{5}$. Therefore our two critical numbers will be $x = \frac{8}{5}$ and $x = 0$. Now we will put these on our table:

0 8/5

I will use test points x = –1, 1, and 2. After putting these into the **DERIVATIVE** we will get:

+	–	+

0 8/5

So we see it is decreasing on $\left(0,\ \frac{8}{5}\right)$ and increasing on $(-\infty,0)\cup\left(\frac{8}{5},\infty\right)$. The local maximum is at (0, 0) and the local minimum is at $\left(\frac{8}{5}, -\frac{36}{5}\left(\frac{8}{5}\right)^{\frac{2}{3}}\right)$ or $\left(\frac{8}{5}, -9.85\right)$. There is no absolute max and min because the endpoints are increasing or decreasing forever.