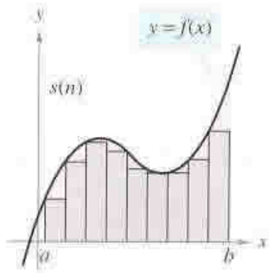
5.2 Approximating Areas with Limits of Finite Sums

This chapter deals with finding the area under curves, which is what integrals will do. In this section, we will be finding the area of rectangular strips, and then adding all of these together. There are two ways to add rectangles together:

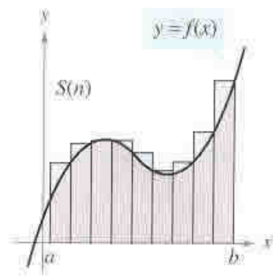
Upper and Lower Sums

The diagrams below explain the difference between inscribed and circumscribed rectangles. It also shows the difference between the upper and lower sum. In the drawing S(n) represents the sum of the individual areas.



Area of inscribed rectangles is less than area of region.

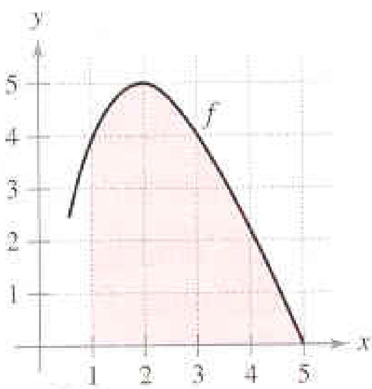
Lower Sum



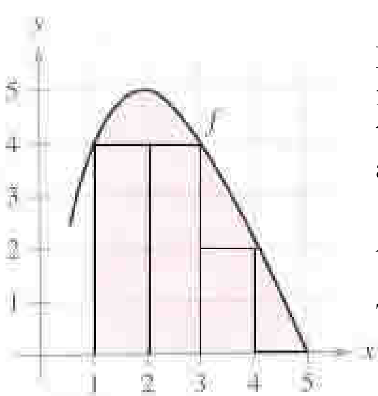
Area of circumscribed rectangles is greater than area of region.

Upper Sum

EXAMPLE: Estimate the area under the curve on [1, 5] by using upper and lower sums. Use rectangles of width 1.



First we need to draw the lower sum. Then we will find the area of each rectangle:

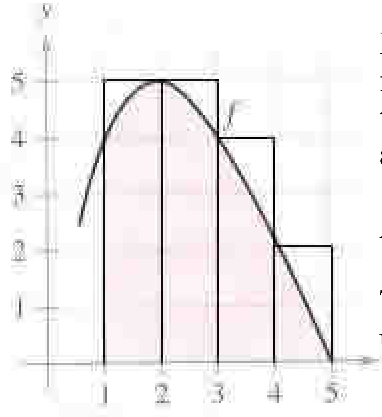


From the drawing we have 4 rectangles. Each one has a width of 1 unit. For the first two rectangles, the height is 4. The third rectangle has a height of 2. The last triangle has a height of 0. The area of a rectangle is A = L*W. We will find the area of each rectangle individually and then add them all together:

$A = 4(1) + 4(1) + 2(1) + 0(1) = 10$

The above shows the areas of each being calculated. The total area is 10 units.

Now we will draw the upper sum. Then we will find the area of each rectangle.

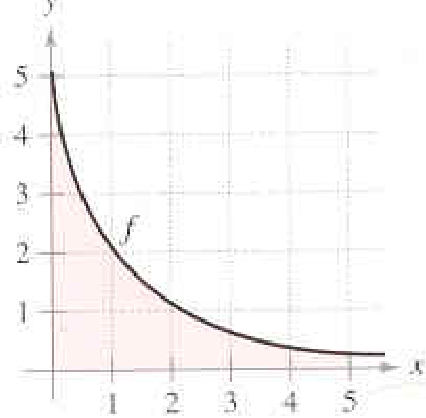


From the drawing we have 4 rectangles. Each one has a width of 1 unit. For the first two rectangles, the height is 5. The third rectangle has a height of 4. The last triangle has a height of 2. The area of a rectangle is A = L*W. We will find the area of each rectangle individually and then add them all together:

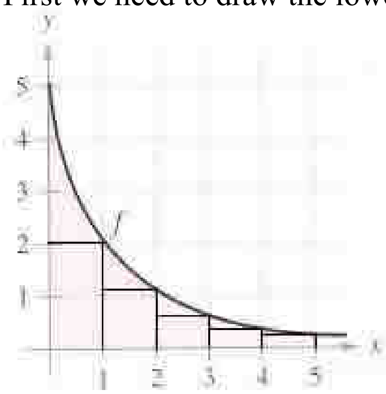
A = 5(1) + 5(1) + 4(1) + 2(1) = 16

The exact area under the curve will be between 10 and 16 since we have found the upper and lower sum.

EXAMPLE: Estimate the area under the curve on [0, 5] by using upper and lower sums. Use rectangles of width 1.



First we need to draw the lower sum. Then we will find the area of each rectangle:

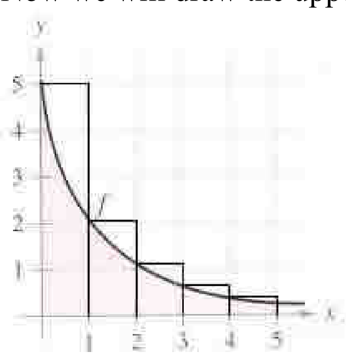


From the drawing we have 5 rectangles. Each one has a width of 1 unit. Looking at the graph the first rectangle has a height of 2, The second triangle has a height of 1. The next heights are approximately 2/3, 1/2, and 1/3.

A = 2(1) + 1(1) + (1/2)(1) + (2/3)(1) + (1/3)(1)

The above shows the areas of each being calculated. The total area is 9/2 units.

Now we will draw the upper sum. Then we will find the area of each rectangle.



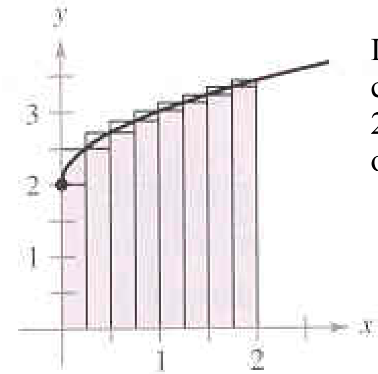
From the drawing we have 5 rectangles. Each one has a width of 1 unit. Looking at the graph the first rectangle has a height of 5, The second triangle has a height of 2. The next heights are approximately 1 then 2/3 then about 1/2.

A = 5(1) + 2(1) + 1(1) + (2/3)(1) + (1/2)(1)

The above shows the areas of each being calculated. The total area is 55/6.

The exact area under the curve will be between 4.5 and 55/6 since we have found the upper and lower sum.

EXAMPLE: Use upper and lower sums to approximate the area under the curve $y = \sqrt{x} + 2$ and above the x-axis on the interval [0, 2]. Use 8 subintervals as shown below:



If we take 2 and divide it by 8 then the width of each rectangle is 0.25. Now we can label each x coordinate . We start with 0 and keep adding 0.25 until we reach 2. Each coordinate is: 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75. Now let's make a table of values:

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2.0
y	2	2.5	2.71	2.87	3	3.12	3.22	3.32	3.41

We will start with the lower sum. The first rectangle will have a height of 2 according to the picture. The second height is 2.5 when x = 0.25. Each sequential height is read off of the table above. Now we can calculate the area. Notice we do not have a height of 2 since we are doing lower sums.

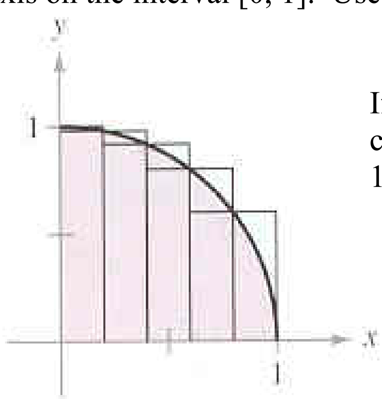
A = 2(0.25) + 2.5(0.25) + 2.71(0.25) + 2.87(0.25)+ 3(0.25) + 3.12(0.25)+ 3.22(0.25) + 3.32(0.25) = 5.67.

Now we will do the upper sum. Now the first rectangle has a height of 2.5.

A = 2.5(0.25) + 2.71(0.25) + 2.87(0.25)+ 3(0.25) + 3.12(0.25)+ 3.22(0.25) + 3.32(0.25) + 3.41(0.25) = 6.04.

The exact area should be between 5.67 and 6.04.

EXAMPLE: Use upper and lower sums to approximate the area under the curve $y = \sqrt{1 - x^2}$ and above the x-axis on the interval [0, 1]. Use 5 subintervals as shown below:



If we take 1 and divide it by 5 then the width of each rectangle is 0.2. Now we can label each x coordinate . We start with 0 and keep adding 0.2 until we reach 1. Each coordinate is: 0.2, 0.4, 0.6, 0.8, 1. Now let's make a table of values:

x	0	0.2	0.4	0.6	0.8	1
y	1	0.98	0.92	0.8	0.6	0

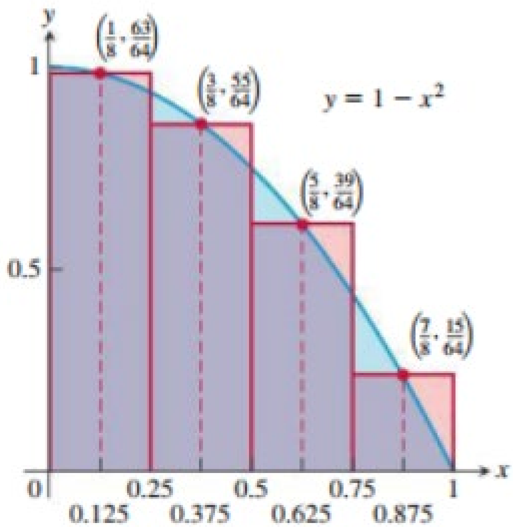
We will start with the lower sum. The first rectangle will have a height of 0.98 according to the picture. The second height is 0.92 when x = 0.4. Each sequential height is read off of the table above. Now we can calculate the area.

A = 0.98(0.2) + 0.92(0.2) + 0.8(0.2) + 0.6(0.2) + 0(0.2) = 0.66

Now we will do the upper sum. Now the first rectangle has a height of 1.

A = 1(0.2) + 0.98(0.2) + 0.92(0.2) + 0.8(0.2) + 0.6(0.2) = 0.86

EXAMPLE: Use rectangles each of whose height is determined by $y = 1 - x^2$ at the midpoint of the rectangle's base (*the midpoint rule*) and estimate the area under the curve $y = 1 - x^2$ by using four rectangles between the x-values of 0 and 1.



If we take 1 and divide it by 4, then the width of each rectangle is 0.25. Now we can label the x-axis. Next we want to label the midpoint of each rectangle. To find the midpoint of the first rectangle we will do the following calculation: (0 + 0.25) / 2, and the result is 0.125. For the second rectangle we do the following calculation: (0.25 + 0.5) / 2, and the result is 0.375. For the third rectangle we do the following calculation: (0.5 + 0.75) / 2 = 0.625. Finally for the fourth rectangle we do the following calculation: (0.75 + 1) / 2, and the result is 0.875. Next we need to make a table of values to find the height at each midpoint:

x	0.125	0.375	0.625	0.875
y	0.984	0.859	0.609	0.234

It is now time to find the area. To find the area of each rectangle, multiply the width (0.25) by each of the different heights:

A = 0.984(0.25) + 0.859(0.25) + 0.609(0.25) + 0.234(0.25) = 0.674

Therefore, using the midpoint rule the area under $y = 1 - x^2$ between 0 and 1 is approximately 0.674.

Now we want to get more accurate when we find the area under the curve. In order to do this we will breaking up the area into many (infinite number) of small strips and then adding them all together. This will involve summations, or a series of sums that involve the sigma notation:

Sigma Notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

EXAMPLE: Evaluate: $\sum_{i=1}^{4} 3i-2$.

To do this we will start i at 1. We will put 1 into our equation. Then we will make i = 2 and put this in the equation, and so on until we reach i = 4. We will take all the answers we got and add them together:

When i = 1, our expression is 3(1) – 2 = 1.
When i = 2, our expression is 3(2) – 2 = 4.
When i = 3, our expression is 3(3) – 2 = 7.
When i = 4, our expression is 3(4) – 2 =10.

To get the answer we will add: 1 + 4 + 7 + 10 = 22. So, $\sum_{i=1}^{4} 3i-2=22$.

EXAMPLE: Evaluate: $\sum_{k=1}^{4} (-1)^k \cos(k\pi)$.

To do this we will start k at 1. We will put 1 into our equation. Then we will make k = 2 and put this in the equation, and so on until we reach k = 4. We will take all the answer we got and add them together:

When k = 1 our expression is $(-1)^1 \cos(1\pi)=(-1)(-1)=1$.
When k = 2, our expression is $(-1)^2 \cos(2\pi)=(1)(1)=1$.
When k = 3, our expression is $(-1)^3 \cos(3\pi)=(-1)(-1)=1$.
When k = 4, our expression is $(-1)^4 \cos(4\pi)=(1)(1)=1$.

To get the answer we will add: 1 + 1 + 1 +1 = 4. So, $\sum_{k=1}^{4} (-1)^k \cos(k\pi)=4$.

EXAMPLE: Use sigma notation to write the sum: $\frac{5}{1+1}+\frac{5}{1+2}+\frac{5}{1+3}+...+\frac{5}{1+15}$.

We need to come up with the formula that describes this sum. First we notice that there will always be a 5 in the numerator of the fraction. Also there is part of this fraction that starts at 1 and goes to 15. Therefore we know that i must start at 1 and go to 15. Also in the numerator there is a 1 that does not change. So we are ready to write our formula: $\sum_{i=1}^{15} \frac{5}{1+i}$. So we will write: $\frac{5}{1+1}+\frac{5}{1+2}+\frac{5}{1+3}+...+\frac{5}{1+15}=\sum_{i=1}^{15} \frac{5}{1+i}$.

EXAMPLE: Use sigma notation to write the sum: $-\frac{1}{7}+\frac{2}{7}-\frac{3}{7}+\frac{4}{7}-\frac{5}{7}+\frac{6}{7}-1$.

For this one we see the denominator is always 7. We have an alternating sign, so this can be written as $(-1)^k$ since the first term needs to be negative. The top is changing one by one, so the formula is: $\sum_{1}^{7}(-1)^k \cdot \frac{k}{7}$.

Algebra Rules for Finite Sums

1.) $\sum_{i=1}^{n} k \cdot a_i = k\sum_{i=1}^{n} a_i$ Any constants we can write in front of the summation symbol.

2.) $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$ We can split up the summation into two separate ones.

Summation Formulas

1.) $\sum_{i=1}^{n} c = c \cdot n$

2.) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$

3.) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$

4.) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4+2n^3+n^2}{4}$

EXAMPLE: Evaluate: $\sum_{i=1}^{15} 2i-3$ using properties of summation and the summation formulas.

We are not going to do this the same way as the ones we did at the beginning of this section. That would take much too long since we have to do 15 different calculations and then add them together. Instead we are going to use the summation properties to break this up: $2\sum_{i=1}^{15} i - \sum_{i=1}^{15} 3$. Now we will use summation formulas. In the first term we will replace $\sum_{i=1}^{15} i$ with $\frac{15(15+1)}{2}$ using formula #2 since that is the corresponding formula and n = 15. We will replace the second term $\sum_{i=1}^{15} 3$ with 3(15) using formula #1. So our problem now becomes:

$2\cdot\dfrac{15(15+1)}{2}-3(15)$. We can cancel the twos and simplify: $15(16)-3(15)=240-45=195$. Therefore, $\sum_{i=1}^{15}2i-3=195$.

EXAMPLE: Evaluate: $\sum_{i=1}^{10}i^2-2i^3$ using properties of summation and the summation formulas.

We will use the summation properties to break this up: $\sum_{i=1}^{10}i^2-2\sum_{i=1}^{10}i^3$. Now we will use summation formulas. In the first term we will replace $\sum_{i=1}^{10}i^2$ with $\dfrac{10(10+1)(2(10)+1)}{6}$ using formula #3. We will replace the second term $\sum_{i=1}^{10}i^3$ with $\dfrac{10^2(10+1)^2}{4}$ using formula #4. Now our problem becomes:

$\dfrac{10(10+1)(2(10)+1)}{6}-2\cdot\dfrac{10^2(10+1)^2}{4}$. This simplifies to: $\dfrac{10(11)(21)}{6}-\dfrac{2(100)(121)}{4}=385-6050=-5665$.

EXAMPLE: Find the limit: $\lim_{n\to\infty}\dfrac{1}{n^2}\left[\sum_{i=1}^{n}i\right]$.

First we need to put in the summation formula inside the brackets: $\lim_{n\to\infty}\dfrac{1}{n^2}\left[\dfrac{n(n+1)}{2}\right]$. Next we multiply to get one fraction: $\lim_{n\to\infty}\dfrac{n^2+n}{2n^2}$. Now divide each term in the numerator by the term in the denominator:

$\lim_{n\to\infty}\dfrac{n^2}{2n^2}+\dfrac{n}{2n^2}$. This reduces to: $\lim_{n\to\infty}\dfrac{1}{2}+\dfrac{1}{2n}$. When we take the limit the last term will go to zero, so $\lim_{n\to\infty}\dfrac{n^2+n}{2n^2}=\dfrac{1}{2}$.

EXAMPLE: Find the limit: $\lim_{n\to\infty}\dfrac{64}{n^3}\left[\dfrac{n(n+1)(2n+1)}{6}\right]$.

First we can multiply the numerator and reduce $\dfrac{64}{6}$: $\lim_{n\to\infty}\dfrac{32}{n^3}\left[\dfrac{2n^3+3n^2+n}{3}\right]$. We can combine this into one fraction: $\lim_{n\to\infty}\dfrac{64n^3+96n^2+32n}{3n^3}$. Now I will divide each term in the numerator by the term in the denominator:

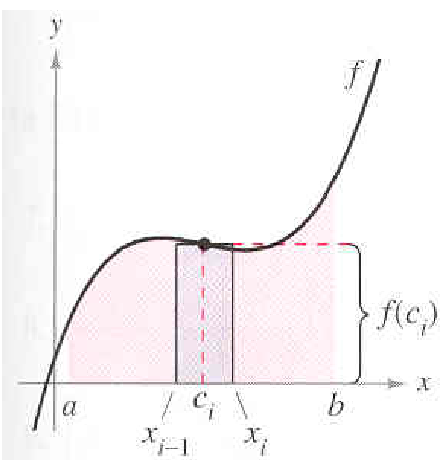
$\lim_{n\to\infty} \frac{64n^3}{3n^3} + \frac{96n^2}{3n^3} + \frac{32n}{3n^3}$. This reduces to $\lim_{n\to\infty} \frac{64}{3} + \frac{32}{n} + \frac{32}{3n^2}$. When we take the limit the second two terms will go to zero, so $\lim_{n\to\infty} \frac{32}{n^3}\left[\frac{2n^3 + 3n^2 + n}{3}\right] = \frac{64}{3}$.

EXAMPLE: Find the limit: $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$.

We have a limit of a summation here. First let's multiply the two fraction together to get: $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n^2}\right)$. Now pull out the $\frac{4}{n^2}$ from inside the summation: $\lim_{n\to\infty} \frac{4}{n^2} \sum_{i=1}^{n} i$. We will use a summation formula for $\sum_{i=1}^{n} i$, which is $\frac{n(n+1)}{2}$. Our problem now becomes: $\lim_{n\to\infty} \frac{4}{n^2} \frac{n(n+1)}{2}$. This reduces to: $\lim_{n\to\infty} \frac{2n+2}{n}$. Now divide it into two fractions: $\lim_{n\to\infty} 2 + \frac{2}{n}$. The second term goes to zero as *n* approaches infinity, so $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = 2$.

Using the Limit Process to Find the Area Under a Curve

In order to find the area under a curve, we are going to find the area of several rectangles and then add them all together to get the total area. Let's first look at one of those rectangles and define some variables:



The width of the rectangle is found by taking subtracting a from b and then dividing by the total number of subintervals. If the number of subintervals is n, then the width of each box is $\Delta x = \frac{b-a}{n}$. The height of each rectangle is found by using the function f. The height of the rectangle at some x value, c_i, will be $f(c_i)$. So the area of this rectangle will be $A = f(c_i) \cdot \Delta x$. To get an exact area we need to add an infinite number of these rectangles together between a and b. If we let the number of subintervals be n, then we want n to go to infinity. Then we get our formula for area.

The area under a curve and above the x-axis is: $\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta x$ where $\Delta x = \frac{b-a}{n}$ and $c_i = a + \Delta x i$.

The formula for c_i is found by starting with a and adding the Δx i times. Alternatively, some texts use *k* instead of *i*. In that case, $\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x$ where $\Delta x = \frac{b-a}{n}$ and $c_k = a + \Delta x k$.

EXAMPLE: Find the formula for the Riemann sum obtained by dividing the interval [a, b] into *n* equal subintervals and using the right endpoint for each c_k. Then take the limit of these sums as $n \to \infty$ to calculate the area under the curve $f(x) = 1 + x^2$ over [0, 3].

Since this function is increasing, in order to use upper sums then the right endpoints of our rectangles will determine our heights. Our interval tells us that *a* is 0 and *b* is 3. Then we know the Δx by using the formula:

$\Delta x = \frac{3-0}{n}$, so $\Delta x = \frac{3}{n}$. To find the c_k we will use its formula: $c_k = 0 + \left(\frac{3}{n}\right)k$, so $c_k = \frac{3k}{n}$. Now we will substitute into the formula $\lim_{n\to\infty} f(c_k)\cdot\Delta x$ to get: $\lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{3k}{n}\right)\cdot\frac{3}{n}$, which is the correct formula for the upper sum. We can put the $\frac{3}{n}$ in front of the summation: $\lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n} f\left(\frac{3k}{n}\right)$. To find $f\left(\frac{3k}{n}\right)$ we will put $\frac{3k}{n}$ into $f(x) = 1 + x^2$ for x. This will give us $y = 1 + \left(\frac{3k}{n}\right)^2$. Now our problem becomes $\lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}\left(1+\left(\frac{3k}{n}\right)^2\right)$. We can break up this summation into two separate ones: $\lim_{n\to\infty}\frac{3}{n}\left[\sum_{k=1}^{n}1+\sum_{k=1}^{n}\frac{9k^2}{n^2}\right]$ This further simplifies to: $\lim_{n\to\infty}\frac{3}{n}\left[\sum_{k=1}^{n}1+\frac{9}{n^2}\sum_{k=1}^{n}k^2\right]$. Now we will now put in summation formulas where I have *k*: $\lim_{n\to\infty}\frac{3}{n}\left[n(1)+\frac{9}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$. We can multiply the top part and reduce: $\lim_{n\to\infty}\frac{3}{n}\left[n(1)+\frac{3}{n^2}\cdot\frac{2n^3+3n^2+n}{2}\right]$. Now multiply: $\lim_{n\to\infty}\frac{3}{n}\left[n(1)+\frac{6n^3+9n^2+3n}{2n^2}\right]$. Now distribute the $\frac{3}{n}$: $\lim_{n\to\infty} 3+\frac{18n^3+27n^2+9n}{2n^3}$. We will break up the second fraction: $\lim_{n\to\infty} 3+\frac{18n^3}{2n^3}+\frac{27n^2}{2n^3}+\frac{9n}{2n^3}$. This reduces to: $\lim_{n\to\infty} 3+9+\frac{27}{2n}+\frac{9}{2n^2}$. Now take the limit, and our answer is 12.

EXAMPLE: Find the formula for the Riemann sum obtained by dividing the interval [a, b] into *n* equal subintervals and using the right endpoint for each c_k. Then take the limit of these sums as $n \to \infty$ to calculate the area under the curve $f(x) = 5x$ over [1, 3].

Since this function is increasing, in order to use upper sums then the right endpoints of our rectangles will determine our heights. Our interval tells us that *a* is 1 and *b* is 3. Then we know the Δx by using the formula: $\Delta x = \frac{3-1}{n}$, so $\Delta x = \frac{2}{n}$. To find the c_k we will use its formula: $c_k = 1 + \left(\frac{2}{n}\right)k$, so $c_k = 1 + \frac{2k}{n}$. Now we will substitute into the formula $\lim_{x\to\infty} f(c_k)\cdot\Delta x$ to get: $\lim_{n\to\infty}\sum_{k=1}^{n} f\left(1+\frac{2k}{n}\right)\cdot\frac{2}{n}$. To find $f\left(1+\frac{2k}{n}\right)$ we will put $1+\frac{2k}{n}$ in for x in $f(x) = 5x$: $f\left(1+\frac{2k}{n}\right) = 5\left(1+\frac{2k}{n}\right)$. So our formula now is $\lim_{n\to\infty}\sum_{k=1}^{n} 5\left(1+\frac{2k}{n}\right)\cdot\frac{2}{n}$ which will give the formula for the Riemann sum of $f(x) = 5x$ over [1, 3]. This becomes $\lim_{n\to\infty}\frac{10}{n}\sum_{k=1}^{n}\left(1+\frac{2k}{n}\right)$. We can break up this summation into two separate ones: $\lim_{n\to\infty}\frac{10}{n}\left[\sum_{k=1}^{n}1+\frac{2}{n}\sum_{k=1}^{n}k\right]$. Now we will put in summation formulas for the summations involving and *k*: $\lim_{n\to\infty}\frac{10}{n}\left[1\cdot n+\frac{2}{n}\cdot\frac{n^2+n}{2}\right]$. This simplifies to: $\lim_{n\to\infty}\frac{10}{n}\left[n+\frac{n^2+n}{n}\right]$. Now distribute the $\frac{10}{n}$: $\lim_{n\to\infty}\left(10+\frac{10n^2+10n}{n^2}\right)$. We will break up the fractions and reduce: $\lim_{n\to\infty} 10+10+\frac{2}{n}$, which simplifies to $\lim_{n\to\infty} 20+\frac{2}{n}$. Therefore we can say that the sigma notation $\sum_{k=1}^{n} 5\left(1+\frac{2k}{n}\right)\cdot\frac{2}{n}$ simplifies to $20+\frac{2}{n}$.

Now take the limit: $\lim_{n\to\infty} 20+\frac{2}{n}=20+0=4$.

So our area between the curve $f(x)=5x$ and the x-axis on [1, 3] is 20.

EXAMPLE: Find the formula for the Riemann sum obtained by dividing the interval [a, b] into *n* equal subintervals and using the right endpoint for each c_k. Then take the limit of these sums as $n\to\infty$ to calculate the area under the curve $f(x)=x+x^3$ over [0, 1].

Since this function is increasing, in order to use upper sums then the right endpoints of our rectangles will determine our heights. Our interval tells us that *a* is 1 and *b* is 3. Then we know the Δx by using the formula: $\Delta x=\frac{1-0}{n}$, so $\Delta x=\frac{1}{n}$. To find the c_k we will use its formula: $c_k=0+\left(\frac{1}{n}\right)k$, so $c_k=\frac{k}{n}$. Now we will substitute into the formula $\lim_{x\to\infty} f(c_k)\cdot\Delta x$ to get: $\lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n}$. To find $f\left(\frac{k}{n}\right)$ we will put $\frac{k}{n}$ in for x in $f(x)=x+x^3$: $f\left(\frac{k}{n}\right)=\frac{k}{n}+\left(\frac{k}{n}\right)^3$. So our formula now is $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}+\frac{k^3}{n^3}\right)\cdot\frac{1}{n}$ which will give the formula for the Riemann sum of $f(x)=x+x^3$ over [0, 1]. This becomes $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}+\frac{k^3}{n^3}\right)$. We can break up this summation into two separate ones: $\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{n}\sum_{k=1}^{n} k+\frac{1}{n^3}\sum_{k=1}^{n} k^3\right]$. Now use summation formulas:

$\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{n}\cdot\frac{n^2+n}{2}+\frac{1}{n^3}\cdot\frac{n^4+2n^3+n^2}{4}\right]$. This simplifies to: $\lim_{n\to\infty}\frac{1}{n}\left[\frac{n^2+n}{2n}+\frac{n^4+2n^3+n^2}{4n^3}\right]$. Now distribute $\frac{1}{n}$:

$\lim_{n\to\infty}\left(\frac{n^2+n}{2n^2}+\frac{n^4+2n^3+n^2}{4n^4}\right)$. We will break up the fractions and reduce: $\lim_{n\to\infty}\frac{1}{2}+\frac{1}{2n}+\frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}$, which simplifies to $\lim_{n\to\infty}\frac{3}{4}+\frac{4n+1}{4n^2}$. Therefore we can say that the sigma notation $\sum_{k=1}^{n}\left(\frac{k}{n}+\frac{k^3}{n^3}\right)\cdot\frac{1}{n}$ simplifies to $\frac{3}{4}+\frac{4n+1}{4n^2}$. Now take the limit: $\lim_{n\to\infty}\frac{3}{4}+\frac{4n+1}{4n^2}=\lim_{n\to\infty}\frac{3}{4}+\frac{1}{n}+\frac{1}{4n^2}=\frac{3}{4}+0+0=\frac{3}{4}$.

So our area between the curve $f(x)=x+x^3$ and the x-axis on [0, 0] is 3/4.