5.5 Substitution with Indefinite Integrals

You may have a complicated equation that needs to be simplified. This is accomplished by substituting a variable, getting the antiderivative, and then substituting back to the original variable. What we are doing here is the reverse of the chain rule. The chain rule says this: if we start with $y = f(u)$ then $y' = f'(u) \cdot u'$. Therefore if we start with the derivative and take the antiderivative we can get back to our original function. In our case we will let $u = g(x)$.

Integration by Substitution:

Let g be a function whose range is in an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then:

$\int f(g(x)) \cdot g'(x)dx = F(g(x)) + C$.

If $u = g(x)$ then $du = g'(x)dx$ and $\int f(u)du = F(u) + C$.

Like in the chain rule, $g(x)$ is usually an 'inside' function that needs to be identified.

How to Integrate by Substitution:

1.) Let $u = g(x)$.

2.) Take the derivative of both sides to get $du = g'(x)\ dx$.

3.) Solve for dx and substitute this and u into the equation.

4.) Take the antiderivative of u.

5.) Substitute back in the $g(x)$ for u.

EXAMPLE: Integrate by substitution by using the given substitution to reduce the integral to standard form: $\int (x^2 - 9)^3 (2x)\ dx$, $u = x^2 - 9$.

Let's follow the 5 steps to integrate this:

1.) We are given $u = x^2 - 9$. Now we want to take the derivative of both sides.

2.) First we have $\frac{du}{dx} = 2x$. This will give us: $du = 2x\ dx$.

3.) Solving for dx we get $dx = \frac{du}{2x}$. Now we will substitute this for dx and we will substitute a u for $x^2 - 9$.

Now we have: $\int u^3 (2x) \frac{du}{2x}$. The 2x will cancel, leaving you with: $\int u^3 du$. You should never have any x terms left. Everything should only have a u in it since we are taking the derivative with respect to u.

4.) $\int u^3 du = \frac{u^4}{4} + C$

5.) $\int (x^2 - 9)^3 (2x)\ dx = \frac{(x^2 - 9)^4}{4} + C$.

EXAMPLE: Integrate by substitution by using the given substitution to reduce the integral to standard form:
$\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right)\ dx$, $u = \frac{1}{x}$.

Let's follow the 5 steps to integrate this:

1.) We are given $u = \frac{1}{x} = x^{-1}$. Now we want to take the derivative of both sides.

2.) First we have $\frac{du}{dx} = -x^{-2}$. This will give us: $du = -\frac{1}{x^2}\ dx$.

3.) Solving for dx we get $dx = -x^2 du$. Now we will substitute this for dx and we will substitute a u for $\frac{1}{x}$:

$\int \frac{1}{x^2} \cos^2(u)\ \cdot -x^2 du$. We can simplify to get: $-\int \cos^2 u\ du$.

In order to do the antiderivative it would be best to use the power-reducing identity: $\cos^2 \theta = \frac{1 - \cos 2\theta}{2}$. So our problem can be rewritten as: $-\int \frac{1 - \cos 2u}{2} du$. Then we can split the fraction into: $-\int \frac{1}{2} - \frac{\cos 2u}{2} du$.
Finally we are ready for step 4 to find the antiderivative:

4.) $-\int \frac{1}{2} - \frac{\cos 2u}{2} du \quad = \quad -\frac{u}{2} + \frac{1}{4}\sin 2u + C$.

5.) $\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right)\ dx \quad = \quad -\frac{\frac{1}{x}}{2} + \frac{1}{4}\sin\left(2 \cdot \frac{1}{x}\right) + C$. Simplifying and gives us the final answer:
$-\frac{1}{2x} + \frac{1}{4}\sin\left(\frac{2}{x}\right) + C$

EXAMPLE: Integrate by substitution: $\int \frac{x^2}{\sqrt{16-x^3}}\ dx$.

First we will let $u = 16 - x^3$. Then $du = -3x^2\ dx$. Solving for dx we get $dx = \frac{du}{-3x^2}$. Now we will substitute this for dx and we will substitute a u for $16 - x^3$. Now we have $\int \frac{x^2}{\sqrt{u}} \cdot \frac{du}{-3x^2}$. We get $\int \frac{du}{-3\sqrt{u}}$, or $-\frac{1}{3}\int u^{-\frac{1}{2}} du$. The antiderivative is $-\frac{1}{3}\int u^{\frac{1}{2}} du = -\frac{1}{3} \cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$, which simplifies to $-\frac{2}{3}\sqrt{u} + C$. Now substitute back in the x: $\int \frac{x^2}{(16-x^3)^2}\ dx = -\frac{2}{3}\sqrt{16-x^3} + C$.

EXAMPLE: Integrate by substitution: $\int x^3\sqrt{1+\frac{x^4}{8}}\ dx$.

First we will let $u = 1 + \frac{x^4}{8}$. Then $du = \frac{1}{2}x^3\ dx$. Solving for dx we get $dx = \frac{2du}{x^3}$. Now we will substitute this for dx and we will substitute a u for $1 + \frac{x^4}{8}$. Now we have $\int x^3\sqrt{u} \cdot \frac{2du}{x^3}$. We get $2\int \sqrt{u}\ du$, or $2\int u^{\frac{1}{2}} du$.

The antiderivative is $2\int u^{\frac{1}{2}} du = \frac{2u^{\frac{3}{2}}}{\frac{3}{2}} + C$, which simplifies to $\frac{4}{3}u^{\frac{3}{2}} + C$. Now substitute back in the x:

$$\int x^3\sqrt{1+\frac{x^4}{8}}\ dx = \frac{4}{3}\left(1+\frac{x^4}{8}\right)^{\frac{3}{2}} + C.$$

EXAMPLE: Integrate by substitution: $\int \frac{\sin(2x)}{\cos^3(2x)}\ dx$.

First we will let $u = \cos(2x)$. Then $du = -2\sin(2x)\ dx$. Solving for dx we get $dx = \frac{du}{-2\sin(2x)}$. Now we will substitute this for dx and we will substitute a u for $\cos(2x)$. Now we have. We get $\int \frac{\sin(2x)}{u^3} \cdot \frac{du}{-2\sin(2x)}$ which simplifies to $-\frac{1}{2}\int \frac{du}{u^3}$, or $-\frac{1}{2}\int u^{-3} du$. The antiderivative is $-\frac{1}{2}\int u^{-3} du = -\frac{1}{2} \cdot \frac{u^{-2}}{-2} + C$, which simplifies to $\frac{1}{4u^2} + C$. Now substitute back in the x: $\int \frac{\sin(2x)}{\cos^3(2x)}\ dx = \frac{1}{4\cos^2(2x)} + C$.

EXAMPLE: Integrate by substitution: $\int \sec^2(1-x)\tan^7(1-x)\ dx$.

First we will let $u=\tan(1-x)$. Then $du=-\sec^2(1-x)dx$. Solving for dx we get $dx=\dfrac{-du}{\sec^2(1-x)}$. Now we will substitute this for dx and we will substitute a u for $\tan(1-x)$. Now we have $\int \sec^2(1-x)u^7\cdot\dfrac{-du}{\sec^2(1-x)}$.

We get $\int -u^7du$. The antiderivative is $\dfrac{-u^8}{8}+C$. Now substitute back in the x: $\int \sec^2(1-x)\tan^7(1-x)\ dx$ = $\dfrac{-\tan^8(1-x)}{8}+C$.

EXAMPLE: Integrate by substitution: $\int \dfrac{1}{x^3}\sin\left(\dfrac{1}{x^2}-7\right)\ dx$.

First we will let $u=\dfrac{1}{x^2}-7$. Then $du=-2x^{-3}dx$. This can be rewritten as $du=\dfrac{-2}{x^3}dx$ Solving for dx we get $dx=\dfrac{-x^3}{2}du$. Now we will substitute this for dx and we will substitute a u for $\dfrac{1}{x^2}-7$. Now we have $\int \dfrac{1}{x^3}\sin(u)\ \cdot\dfrac{-x^3}{2}du$. We get $-\dfrac{1}{2}\int \sin(u)\ du$. The antiderivative is $-\dfrac{1}{2}\cdot -\cos(u)+C$. Now substitute back in the x: $\int \dfrac{1}{x^3}\sin\left(\dfrac{1}{x^2}-7\right)\ dx\ =\ \dfrac{1}{2}\cos\left(\dfrac{1}{x^2}-7\right)+C$.

Integrals Involving Inverse Trigonometric Functions with Substitutions

$$\int \frac{du}{\sqrt{a^2-u^2}}=\sin^{-1}\frac{u}{a}+C$$

$$\int \frac{du}{a^2+u^2}=\frac{1}{a}\tan^{-1}\frac{u}{a}+C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{|u|}{a}+C$$

EXAMPLE: Integrate by substitution: $\int \dfrac{4}{1+9x^2}\ dx$.

We can rewrite this problem as: $\int \dfrac{4}{1^2+(3x)^2}\ dx$ Now we are going to solve this by substitution. We are going to let $u=3x$. Then $du=3dx$. Solving for dx we get: $dx=\dfrac{du}{3}$. Now we will make our substitutions:

$\int \frac{4}{1^2+u^2} \cdot \frac{du}{3}$. Simplifying we get: $\frac{4}{3}\int \frac{du}{1^2+u^2}$. This fits the second formula above if a is 1: $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$. Integrating we get: $\frac{4}{3}\cdot\frac{1}{1}\tan^{-1}\frac{u}{1} + C$. Now we put in a 3x for u to get:

$\frac{4}{3}\tan^{-1}(3x) + C$.

EXAMPLE: Find the indefinite integral: $\int \frac{dx}{x^2+4x+13}$.

The bottom cannot be factored. I need to make it look like something squared in order to use my formulas. This involves completing the square. I will first rewrite the bottom as: $(x^2+4x\quad)+13$. I will take the 4 and divide it by two. Then I will square this to get 4. I will add a 4 inside the parenthesis and subtract it from the 13 so that I don't change the equation. You will get: $(x^2+4x+4)+13-4$. Now I will factor what is inside the parenthesis and simplify the numbers outside the parenthesis: $(x+2)^2+9$. So now our problem is: $\int \frac{dx}{(x+2)^2+9}$. We can rewrite this as: $\int \frac{dx}{(x+2)^2+3^2}$. In this problem $u = x+2$. Then $du = dx$. After making our substitutions we get: $\int \frac{dx}{u^2+3^2}$. We will be using $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$ with a = 3. After integrating we get: $\frac{1}{3}\tan^{-1}\frac{u}{3} + C$. Now we put in an $x+2$ for u: $\frac{1}{3}\tan^{-1}\frac{x+2}{3} + C$.

EXAMPLE: Integrate by substitution: $\int \frac{\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^3}{\sqrt{9-x^2}}\,dx$.

We are going to let $u = \sin^{-1}\left(\frac{x}{3}\right)$. Recall $\frac{d}{dx}\left[\sin^{-1} w\right] = \frac{w'}{\sqrt{1-w^2}}$ So here $w = \frac{x}{3}$. Putting it into the formula we have $\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{3}\right)\right] = \frac{1/3}{\sqrt{1-\left(\frac{x}{3}\right)^2}}$. We need to simplify this: $\frac{1}{3\sqrt{1-\frac{x^2}{9}}}$. Now we need common denominators: $\frac{1}{3\sqrt{\frac{9-x^2}{9}}}$. Take the square root of the top and bottom separately: $\frac{1}{3\frac{\sqrt{9-x^2}}{3}}$. This simplifies, so we get: $\frac{du}{dx} = \frac{1}{\sqrt{9-x^2}}$. Solving for *dx* we get $dx = \sqrt{9-x^2}\,du$. Now we will make our substitutions:

$\int \frac{u^3}{\sqrt{9-x^2}} \cdot \sqrt{9-x^2}\,du$. Simplifying we get: $\int u^3 du$. Integration gives us $\frac{u^4}{4}+C$. Finally we replace the u to get our final answer: $\frac{\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^4}{4}+C$

Integration of a Natural Logarithm

Let u be a differentiable function of x. Then:

1.) $\int \frac{1}{x}\ dx = \ln|x| + C$

2.) $\int \frac{1}{u}\ du = \ln|u| + C$ Since $du = u'\ dx$ we can rewrite this as: $\int \frac{u'}{u}\ du = \ln|u| + C$

We have the absolute value symbols here so that any x value will fit the domain of the natural logarithm. Now let's look at some examples:

EXAMPLE: Find the indefinite integral: $\int \frac{x(x+2)}{x^3+3x^2-4}\ dx$.

Using substitution we will let $u = x^3+3x^2-4$. Then $du = 3x^2+6x\ dx$. Solving for dx and after factoring we get: $dx = \frac{du}{3x(x+2)}$. Now we make our substitution: $\int \frac{x(x+2)}{u} \cdot \frac{du}{3x(x+2)}$. This simplifies to: $\frac{1}{3}\int \frac{1}{u}\ du$. When we integrate this we get $\frac{1}{3}\ln|u| + C$. Then we can replace the u with x^3+3x^2-4 and we get: $\frac{1}{3}\ln\left|x^3+3x^2+4\right| + C$ which is our answer.

EXAMPLE: Find the indefinite integral: $\int \frac{3\sec^2 t}{6+3\tan t}\ dt$.

Using substitution we will let $u = 6+3\tan t$. Then $du = 3\sec^2 t\ dt$. Solving for dt we get: $dt = \frac{du}{3\sec^2 t}$. Now we make our substitution: $\int \frac{3\sec^2 t}{u} \cdot \frac{du}{3\sec^2 t}$. This simplifies to: $\int \frac{1}{u}\ du$. When we integrate this we get $\ln|u| + C$. Then we can replace the u with $6+3\tan t$ and we get: $\ln|6+3\tan t| + C$ which is our answer.

EXAMPLE: Find the indefinite integral: $\int \frac{1}{x^{\frac{2}{3}}\left(1+x^{\frac{1}{3}}\right)}\, dx$.

Using substitution we will get $u = 1 + x^{\frac{1}{3}}$. Then $du = \frac{1}{3}x^{-\frac{2}{3}}\, dx$. Solving for dx we get: $dx = 3x^{\frac{2}{3}}du$. Now we make our substitution: $\int \frac{1}{x^{\frac{2}{3}} \cdot u} \cdot 3x^{\frac{2}{3}}\, du$. This simplifies to: $3\int \frac{1}{u}\, du$. When we integrate this we get $3\ln|u| + C$. Then we can replace the u with $1 + x^{\frac{1}{3}}$ and we get: $3\ln\left|1 + x^{\frac{1}{3}}\right| + C$ as our answer.

EXAMPLE: Find the indefinite integral: $\int \tan x\, dx$.

We haven't done this one yet. First let's use identities to write this as: $\int \frac{\sin x}{\cos x}\, dx$. Now we can use substitution to integrate this. We will let $u = \cos x$. Then $du = -\sin x\, dx$. Solving for dx you will get: $dx = -\frac{du}{\sin x}$. Now we make our substitution: $\int \frac{\sin x}{u} \cdot -\frac{du}{\sin x}$. Simplifying will give us: $-\int \frac{1}{u}\, du$. Integrating this will give: $-\ln|u| + C$. Then we can replace the u with $\cos x$ and we get: $-\ln|\cos x| + C$.

We can do a similar process for $\cot x$, $\sec x$, and $\csc x$ to get the following results:

Integrals of the Six Trigonometric Functions

$\int \sin u\, du = -\cos u + C$ $\qquad$ $\int \cos u\, du = \sin u + C$

$\int \tan u\, du = -\ln|\cos u| + C$ $\qquad$ $\int \cot u\, du = \ln|\sin u| + C$

$\int \sec u\, du = \ln|\sec u + \tan u| + C$ $\qquad$ $\int \csc u\, du = -\ln|\csc u + \cot u| + C$

EXAMPLE: Find the indefinite integral: $\int \sec\frac{x}{2}\ dx$.

Using substitution we will get $u = \frac{x}{2}$. Then $du = \frac{1}{2}\ dx$. Solving for dx we get: $dx = 2du$. Now we make our substitution: $\int \sec u \cdot 2du$. This simplifies to: $2\int \sec u\ du$. When we integrate this we get $2\ln\left|\sec u + \tan u\right| + C$. Then we can replace the u with $\frac{x}{2}$ and we get: $2\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + C$.

EXAMPLE: Find the indefinite integral: $\int \csc\theta + \cot\theta\ d\theta$.

We can integrate each thing separately by using the integration formulas: $-\ln\left|\csc\theta + \cot\theta\right| + \ln\left|\sin x\right| + C$

We can use log properties to write this as: $\ln\left|\frac{\sin\theta}{\csc\theta + \cot\theta}\right| + C$. This can also be simplified by using identities:

$\ln\left|\frac{\sin\theta}{\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}}\right| + C$. This equals: $\ln\left|\frac{\sin\theta}{\frac{1+\cos\theta}{\sin\theta}}\right| + C$ which simplifies to: $\ln\left|\frac{\sin^2\theta}{1+\cos\theta}\right| + C$. We can use the identity $\sin^2\theta = 1 - \cos^2\theta$: $\ln\left|\frac{1-\cos^2\theta}{1+\cos\theta}\right| + C$. Now we can factor the numerator: $\ln\left|\frac{(1-\cos\theta)(1+\cos\theta)}{1+\cos\theta}\right| + C$

This simplifies to: $\ln\left|1 - \cos\theta\right| + C$.

EXAMPLE: Find the indefinite integral: $\int (\sin 2\theta)e^{\cos^2\theta}\ d\theta$.

Using substitution we will get $u = \cos^2\theta$. Then $du = -2\cos\theta\sin\theta\ d\theta$. Solving for $d\theta$ we get $d\theta = \frac{du}{-2\cos\theta\sin\theta}$. We can use the trig identity: $\sin 2\theta = 2\sin\theta\cos\theta$. So $d\theta = \frac{du}{-\sin 2\theta}$. Now we make our substitution: $\int (\sin 2\theta)\cdot e^u \cdot \frac{du}{-\sin 2\theta}$. This simplifies to: $-\int e^u\ du$. When we integrate this we get $-e^u + C$. Then we can replace the u with $\cos^2\theta$ and we get: $-e^{\cos^2\theta} + C$.

Change of Variables

In all the problems we did we were able to always cancel out the x terms and just have u. In the next two problems this will not automatically happen so we need to change variables.

EXAMPLE: Integrate by substitution: $\int x(1-x)^4\ dx$.

First we will let $u = 1-x$. Then $du = -dx$. Solving for dx we get $dx = -du$. Now we will substitute this for dx and we will substitute a u for $1-x$. Now we have $-\int x \cdot u^4\ du$. The problem here is that all the x terms did not cancel automatically. What we can do not is to use our formula for u which is $u = 1-x$ and then solve for x. You will get $x = 1-u$. Now we can substitute $1-u$ for x giving you: $-\int (1-u)\cdot u^4\ dx$. After multiplying we will get: $-\int u^4 - u^5\ du$. The antiderivative is $-\frac{u^5}{5} + \frac{u^6}{6} + C$. Now substitute back in the x: $\int x(1-x)^4\ dx = -\frac{(1-x)^5}{5} + \frac{(1-x)^6}{6} + C$.

EXAMPLE: Integrate by substitution: $\int \frac{2x+1}{\sqrt{x+4}}\ dx$.

First we will let $u = x+4$. Then $du = dx$. Now we will substitute this for dx and we will substitute a u for $x+4$. Now we have $\int \frac{2x+1}{\sqrt{u}}\ du$. The problem here is that all the x terms did not cancel automatically. What we can do not is to use our formula for u which is $u = x+4$ and then solve for x. You will get $x = u-4$. Now we can substitute $u-4$ for x giving you: $\int \frac{2(u-4)+1}{\sqrt{u}}\ du$. This simplifies to: $\int \frac{2u-7}{\sqrt{u}}\ du$. We can divide each term in the top by the bottom to get: $\int 2u^{\frac{1}{2}} - 7u^{-\frac{1}{2}}\ du$. The antiderivative is $2\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{7u^{\frac{1}{2}}}{\frac{1}{2}} + C$. This simplifies to: $\frac{4}{3}u^{\frac{3}{2}} - 14u^{\frac{1}{2}} + C$ Now we substitute back in the x: $\int \frac{2x+1}{\sqrt{x+4}}\ dx = \frac{4}{3}(x+4)^{\frac{3}{2}} - 14(x+4)^{\frac{1}{2}} + C$.