5.5 Substitution with Indefinite Integrals

You may have a complicated equation that needs to be simplified. This is accomplished by substituting a variable, getting the antiderivative, and then substituting back to the original variable. What we are doing here is the reverse of the chain rule. The chain rule says this: if we start with $y = f(u)$ then $y' = f'(u) \cdot u'$. Therefore if we start with the derivative and take the antiderivative we can get back to our original function. In our case we will let $u = g(x)$.

Integration by Substitution:

Let g be a function whose range is in an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then:

$\int f(g(x)) \cdot g'(x)dx = F(g(x)) + C$.

If $u = g(x)$ then $du = g'(x)dx$ and $\int f(u)du = F(u) + C$.

Like in the chain rule, $g(x)$ is usually an 'inside' function that needs to be identified.

How to Integrate by Substitution:

1.) Let $u = g(x)$.

2.) Take the derivative of both sides to get $du = g'(x)\ dx$.

3.) Solve for dx and substitute this and u into the equation.

4.) Take the antiderivative of u.

5.) Substitute back in the $g(x)$ for u.

EXAMPLE: Integrate by substitution by using the given substitution to reduce the integral to standard form: $\int (x^2 - 9)^3 (2x)\ dx$, $u = x^2 - 9$.

Let's follow the 5 steps to integrate this:

1.) We are given $u = x^2 - 9$. Now we want to take the derivative of both sides.

2.) First we have $\frac{du}{dx} = 2x$. This will give us: $du = 2x\ dx$.

3.) Solving for dx we get $dx = \frac{du}{2x}$. Now we will substitute this for dx and we will substitute a u for $x^2 - 9$.

Now we have: $\int u^3 (2x) \frac{du}{2x}$. The 2x will cancel, leaving you with: $\int u^3 du$. You should never have any x terms left. Everything should only have a u in it since we are taking the derivative with respect to u.

4.) $\int u^3 du = \frac{u^4}{4} + C$

5.) $\int (x^2 - 9)^3 (2x) \; dx = \frac{(x^2 - 9)^4}{4} + C$.

EXAMPLE: Integrate by substitution by using the given substitution to reduce the integral to standard form:

$\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) \; dx$, $u = \frac{1}{x}$.

Let's follow the 5 steps to integrate this:

1.) We are given $u = \frac{1}{x} = x^{-1}$. Now we want to take the derivative of both sides.

2.) First we have $\frac{du}{dx} = -x^{-2}$. This will give us: $du = -\frac{1}{x^2} \; dx$.

3.) Solving for dx we get $dx = -x^2 du$. Now we will substitute this for dx and we will substitute a u for $\frac{1}{x}$:

$\int \frac{1}{x^2} \cos^2(u) \; \cdot -x^2 du$. We can simplify to get: $-\int \cos^2 u \; du$.

In order to do the antiderivative it would be best to use the power-reducing identity: $\cos^2 \theta = \frac{1 - \cos 2\theta}{2}$. So our problem can be rewritten as: $-\int \frac{1 - \cos 2u}{2} du$. Then we can split the fraction into: $-\int \frac{1}{2} - \frac{\cos 2u}{2} du$.

Finally we are ready for step 4 to find the antiderivative:

4.) $-\int \frac{1}{2} - \frac{\cos 2u}{2} du \;\; = \;\; -\frac{u}{2} + \frac{1}{4} \sin u + C$.

5.) $\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) \; dx \;\; = \;\; -\frac{\frac{1}{x}}{2} + \frac{1}{4} \sin\left(\frac{1}{x}\right) + C$. Simplifying and gives us the final answer: $-\frac{1}{2x} + \frac{1}{4} \sin\left(\frac{1}{x}\right) + C$

EXAMPLE: Integrate by substitution: $\int \frac{x^2}{\sqrt{16-x^3}}\ dx$.

First we will let $u = 16 - x^3$. Then $du = -3x^2\ dx$. Solving for dx we get $dx = \frac{du}{-3x^2}$. Now we will substitute this for dx and we will substitute a u for $16 - x^3$. Now we have $\int \frac{x^2}{\sqrt{u}} \cdot \frac{du}{-3x^2}$. We get $\int \frac{du}{-3\sqrt{u}}$, or $-\frac{1}{3}\int u^{-\frac{1}{2}} du$. The antiderivative is $-\frac{1}{3}\int u^{\frac{1}{2}} du = -\frac{1}{3} \cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$, which simplifies to $-\frac{2}{3}\sqrt{u} + C$. Now substitute back in the x: $\int \frac{x^2}{(16-x^3)^2}\ dx = -\frac{2}{3}\sqrt{16-x^3} + C$.

EXAMPLE: Integrate by substitution: $\int x^3 \sqrt{1+\frac{x^4}{8}}\ dx$.

First we will let $u = 1 + \frac{x^4}{8}$. Then $du = \frac{1}{2}x^3\ dx$. Solving for dx we get $dx = \frac{2du}{x^3}$. Now we will substitute this for dx and we will substitute a u for $1 + \frac{x^4}{8}$. Now we have $\int x^3 \sqrt{u} \cdot \frac{2du}{x^3}$. We get $2\int \sqrt{u}\ du$, or $2\int u^{\frac{1}{2}} du$. The antiderivative is $2\int u^{\frac{1}{2}} du = \frac{2u^{\frac{3}{2}}}{\frac{3}{2}} + C$, which simplifies to $\frac{4}{3}u^{\frac{3}{2}} + C$. Now substitute back in the x:

$$\int x^3 \sqrt{1+\frac{x^4}{8}}\ dx = \frac{4}{3}\left(1+\frac{x^4}{8}\right)^{\frac{3}{2}} + C.$$

EXAMPLE: Integrate by substitution: $\int \frac{\sin(2x)}{\cos^3(2x)}\ dx$.

First we will let $u = \cos(2x)$. Then $du = -2\sin(2x)\ dx$. Solving for dx we get $dx = \frac{du}{-2\sin(2x)}$. Now we will substitute this for dx and we will substitute a u for $\cos(2x)$. Now we have. We get $\int \frac{\sin(2x)}{u^3} \cdot \frac{du}{-2\sin(2x)}$ which simplifies to $-\frac{1}{2}\int \frac{du}{u^3}$, or $-\frac{1}{2}\int u^{-3} du$. The antiderivative is $-\frac{1}{2}\int u^{-3} du = -\frac{1}{2} \cdot \frac{u^{-2}}{-2} + C$, which simplifies to $\frac{1}{4u^2} + C$. Now substitute back in the x: $\int \frac{\sin(2x)}{\cos^3(2x)}\ dx = \frac{1}{4\cos^2(2x)} + C$.

EXAMPLE: Integrate by substitution: $\int \sec^2(1-x)\tan^7(1-x)\ dx$.

First we will let $u=\tan(1-x)$. Then $du=-\sec^2(1-x)dx$. Solving for dx we get $dx=\dfrac{-du}{\sec^2(1-x)}$. Now we will substitute this for dx and we will substitute a u for $\tan(1-x)$. Now we have $\int \sec^2(1-x)u^7\cdot\dfrac{-du}{\sec^2(1-x)}$. We get $\int -u^7du$. The antiderivative is $\dfrac{-u^8}{8}+C$. Now substitute back in the x: $\int \sec^2(1-x)\tan^7(1-x)\ dx$ = $\dfrac{-\tan^8(1-x)}{8}+C$.

EXAMPLE: Integrate by substitution: $\int \dfrac{1}{x^3}\sin\left(\dfrac{1}{x^2}-7\right)\ dx$.

First we will let $u=\dfrac{1}{x^2}-7$. Then $du=-2x^{-3}dx$. This can be rewritten as $du=\dfrac{-2}{x^3}dx$ Solving for dx we get $dx=\dfrac{-x^3}{2}du$. Now we will substitute this for dx and we will substitute a u for $\dfrac{1}{x^2}-7$. Now we have $\int \dfrac{1}{x^3}\sin(u)\ \cdot\dfrac{-x^3}{2}du$. We get $-\dfrac{1}{2}\int \sin(u)\ du$. The antiderivative is $-\dfrac{1}{2}\cdot -\cos(u)+C$. Now substitute back in the x: $\int \dfrac{1}{x^3}\sin\left(\dfrac{1}{x^2}-7\right)\ dx\ =\ \dfrac{1}{2}\cos\left(\dfrac{1}{x^2}-7\right)+C$.

Integrals Involving Inverse Trigonometric Functions with Substitutions

$$\int \frac{du}{\sqrt{a^2-u^2}}=\sin^{-1}\frac{u}{a}+C$$

$$\int \frac{du}{a^2+u^2}=\frac{1}{a}\tan^{-1}\frac{u}{a}+C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{|u|}{a}+C$$

EXAMPLE: Integrate by substitution: $\int \dfrac{4}{1+9x^2}\ dx$.

We can rewrite this problem as: $\int \dfrac{4}{1^2+(3x)^2}\ dx$ Now we are going to solve this by substitution. We are going to let $u=3x$. Then $du=3dx$. Solving for dx we get: $dx=\dfrac{du}{3}$. Now we will make our substitutions:

$\int \frac{4}{1^2+u^2} \cdot \frac{du}{3}$. Simplifying we get: $\frac{4}{3}\int \frac{du}{1^2+u^2}$. This fits the second formula above if a is 1: $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$. Integrating we get: $\frac{4}{3} \cdot \frac{1}{1}\tan^{-1}\frac{u}{1} + C$. Now we put in a 3x for u to get:

$\frac{4}{3}\tan^{-1}(3x) + C$.

EXAMPLE: Find the indefinite integral: $\int \frac{dx}{x^2+4x+13}$.

The bottom cannot be factored. I need to make it look like something squared in order to use my formulas. This involves completing the square. I will first rewrite the bottom as: $(x^2+4x \quad)+13$. I will take the 4 and divide it by two. Then I will square this to get 4. I will add a 4 inside the parenthesis and subtract it from the 13 so that I don't change the equation. You will get: $(x^2+4x+4)+13-4$. Now I will factor what is inside the parenthesis and simplify the numbers outside the parenthesis: $(x+2)^2+9$. So now our problem is: $\int \frac{dx}{(x+2)^2+9}$. We can rewrite this as: $\int \frac{dx}{(x+2)^2+3^2}$. In this problem $u = x+2$. Then $du = dx$. After making our substitutions we get: $\int \frac{dx}{u^2+3^2}$. We will be using $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$ with a = 3. After integrating we get: $\frac{1}{3}\tan^{-1}\frac{u}{3} + C$. Now we put in an $x+2$ for u: $\frac{1}{3}\tan^{-1}\frac{x+2}{3} + C$.

EXAMPLE: Integrate by substitution: $\int \frac{\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^3}{\sqrt{9-x^2}}\,dx$.

We are going to let $u = \sin^{-1}\left(\frac{x}{3}\right)$. Recall $\frac{d}{dx}\left[\sin^{-1} w\right] = \frac{w'}{\sqrt{1-w^2}}$ So here $w = \frac{x}{3}$. Putting it into the formula we have $\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{3}\right)\right] = \frac{1/3}{\sqrt{1-\left(\frac{x}{3}\right)^2}}$. We need to simplify this: $\frac{1}{3\sqrt{1-\frac{x^2}{9}}}$. Now we need common denominators: $\frac{1}{3\sqrt{\frac{9-x^2}{9}}}$. Take the square root of the top and bottom separately: $\frac{1}{3\frac{\sqrt{9-x^2}}{3}}$. This simplifies, so we get: $\frac{du}{dx} = \frac{1}{\sqrt{9-x^2}}$. Solving for *dx* we get $dx = \sqrt{9-x^2}\,du$. Now we will make our substitutions:

$\int \frac{u^3}{\sqrt{9-x^2}} \cdot \sqrt{9-x^2}\,du$. Simplifying we get: $\int u^3 du$. Integration gives us $\frac{u^4}{4}+C$. Finally we replace the *u* to get our final answer: $\frac{\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^4}{4}+C$

Integration of a Natural Logarithm

Let u be a differentiable function of x. Then:

1.) $\int \frac{1}{x}\ dx = \ln|x| + C$

2.) $\int \frac{1}{u}\ du = \ln|u| + C$ Since $du = u'\ dx$ we can rewrite this as: $\int \frac{u'}{u}\ du = \ln|u| + C$

We have the absolute value symbols here so that any x value will fit the domain of the natural logarithm. Now let's look at some examples:

EXAMPLE: Find the indefinite integral: $\int \frac{x(x+2)}{x^3+3x^2-4}\ dx$.

Using substitution we will let $u = x^3+3x^2-4$. Then $du = 3x^2+6x\ dx$. Solving for dx and after factoring we get: $dx = \frac{du}{3x(x+2)}$. Now we make our substitution: $\int \frac{x(x+2)}{u} \cdot \frac{du}{3x(x+2)}$. This simplifies to: $\frac{1}{3}\int \frac{1}{u}\ du$. When we integrate this we get $\frac{1}{3}\ln|u| + C$. Then we can replace the u with x^3+3x^2-4 and we get: $\frac{1}{3}\ln\left|x^3+3x^2+4\right| + C$ which is our answer.

EXAMPLE: Find the indefinite integral: $\int \frac{3\sec^2 t}{6+3\tan t}\ dt$.

Using substitution we will let $u = 6+3\tan t$. Then $du = 3\sec^2 t\ dt$. Solving for dt we get: $dt = \frac{du}{3\sec^2 t}$. Now we make our substitution: $\int \frac{3\sec^2 t}{u} \cdot \frac{du}{3\sec^2 t}$. This simplifies to: $\int \frac{1}{u}\ du$. When we integrate this we get $\ln|u| + C$. Then we can replace the u with $6+3\tan t$ and we get: $\ln|6+3\tan t| + C$ which is our answer.

EXAMPLE: Find the indefinite integral: $\int \frac{1}{x^{\frac{2}{3}}\left(1+x^{\frac{1}{3}}\right)}\,dx$.

Using substitution we will get $u = 1 + x^{\frac{1}{3}}$. Then $du = \frac{1}{3}x^{-\frac{2}{3}}\,dx$. Solving for dx we get: $dx = 3x^{\frac{2}{3}}du$. Now we make our substitution: $\int \frac{1}{x^{\frac{2}{3}} \cdot u} \cdot 3x^{\frac{2}{3}}\,du$. This simplifies to: $3\int \frac{1}{u}\,du$. When we integrate this we get $3\ln|u| + C$. Then we can replace the u with $1 + x^{\frac{1}{3}}$ and we get: $3\ln\left|1 + x^{\frac{1}{3}}\right| + C$ as our answer.

EXAMPLE: Find the indefinite integral: $\int \frac{2x^2 + 7x - 3}{x - 2}\,dx$.

The problem with this one is that if we use the numerator for u and then take the derivative it won't cancel out the denominator. Therefore we need to use long division to break this up into separate expressions:

$$\begin{array}{r}
2x + 11 \\
x - 2 \overline{\smash{\big)}\, 2x^2 + 7x - 3} \\
\underline{2x^2 - 4x} \\
11x - 3 \\
\underline{11x - 22} \\
19
\end{array}$$

First we ask ourselves how many times x goes into $2x^2$. We ignore the -2 at the moment. We know x goes into $2x^2$ an amount of 2x. We then multiply x – 2 by 2x to get $2x^2 - 4x$. We write this on the next line and then we subtract. Then we get $11x - 3$. We now ask how many times does x go into 11x. The answer is 11. We then multiply x – 2 by 11 to get $11x - 22$. We subtract and get a remainder of 19.

So now we know that $\int \frac{2x^2 + 7x - 3}{x - 2}\,dx = \int 2x + 11 + \frac{19}{x - 2}\,dx$. Now we can integrate each part separately.

For the last part, we will let $u = x - 2$. Then $du = dx$. Making the substitution we get: $\int \frac{19}{u}\,du$. Integrating we get $19\ln|u| + C$. Putting this all together and integrating the first two terms we will get:
$x^2 + 11x + 19\ln|x - 2| + C$ as our answer.

EXAMPLE: Find the indefinite integral: $\int \frac{3x^3 - x^2 + x - 2}{x^2 + 2}\,dx$.

The problem with this one is that if we use the numerator for u and then take the derivative it won't cancel out the denominator. Therefore we need to use long division to break this up into separate expressions. Remember that all of the terms must be ordered in descending powers and if there is a missing term you must put in a zero place keeper.

$$\begin{array}{r l} & \quad\quad\;\; 3x-1 \\ x^2+0x+2 & \overline{)\,3x^3-x^2+x-2} \\ & \underline{3x^2+0x^2+6x} \\ & \quad\;\; -x^2-5x-2 \\ & \quad\;\; \underline{-x^2+0x-2} \\ & \quad\quad\quad -5x \end{array}$$

Remember you are always subtracting when doing long division.

So our problem can be rewritten as: $\int \frac{3x^3-x^2+x-2}{x^2+2}\ dx = \int 3x-1-\frac{5x}{x^2+2}\ dx$. Now we can integrate each part separately. For the last part, we will let $u = x^2+2\ dx$. So $du = 2x\ dx$. Solving for dx you get: $dx = \frac{du}{2x}$.

Making the substitution we get: $\int \frac{5x}{u}\cdot\frac{du}{2x}$. Simplifying you will get: $\frac{5}{2}\int \frac{1}{u}\ du$. Integrating we get $\frac{5}{2}\ln|u|+C$. Putting this all together and integrating the first two terms we will get: $\frac{3}{2}x^2-x-\frac{5}{2}\ln|x^2+2|+C$.

EXAMPLE: Find the indefinite integral: $\int \tan x\ dx$.

We haven't done this one yet. First let's use identities to write this as: $\int \frac{\sin x}{\cos x}\ dx$. Now we can use substitution to integrate this. We will let $u = \cos x$. Then $du = -\sin x\ dx$. Solving for dx you will get: $dx = -\frac{du}{\sin x}$. Now we make our substitution: $\int \frac{\sin x}{u}\cdot -\frac{du}{\sin x}$. Simplifying will give us: $-\int \frac{1}{u}\ du$. Integrating this will give: $-\ln|u|+C$. Then we can replace the u with $\cos x$ and we get: $-\ln|\cos x|+C$.

We can do a similar process for $\cot x$, $\sec x$, and $\csc x$ to get the following results:

Integrals of the Six Trigonometric Functions

$\int \sin u\ du = -\cos u + C$

$\int \cos u\ du = \sin u + C$

$\int \tan u\ du = -\ln|\cos u| + C$

$\int \cot u\ du = \ln|\sin u| + C$

$\int \sec u\ du = \ln|\sec u + \tan u| + C$

$\int \csc u\ du = -\ln|\csc u + \cot u| + C$

Now let's look at some more examples…

EXAMPLE: Find the indefinite integral: $\int \sec\frac{x}{2}\ dx$.

Using substitution we will get $u = \frac{x}{2}$. Then $du = \frac{1}{2}\ dx$. Solving for dx we get: $dx = 2du$. Now we make our substitution: $\int \sec u \cdot 2du$. This simplifies to: $2\int \sec u\ du$. When we integrate this we get $2\ln\left|\sec u + \tan u\right| + C$. Then we can replace the u with $\frac{x}{2}$ and we get: $2\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + C$.

EXAMPLE: Find the indefinite integral: $\int \csc\theta + \cot\theta\ d\theta$.

We can integrate each thing separately by using the integration formulas: $-\ln\left|\csc\theta + \cot\theta\right| + \ln\left|\sin x\right| + C$

We can use log properties to write this as: $\ln\left|\frac{\sin\theta}{\csc\theta + \cot\theta}\right| + C$. This can also be simplified by using identities:

$\ln\left|\frac{\sin\theta}{\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}}\right| + C$. This equals: $\ln\left|\frac{\sin\theta}{\frac{1+\cos\theta}{\sin\theta}}\right| + C$ which simplifies to: $\ln\left|\frac{\sin^2\theta}{1+\cos\theta}\right| + C$. We can use the identity $\sin^2\theta = 1 - \cos^2\theta$: $\ln\left|\frac{1-\cos^2\theta}{1+\cos\theta}\right| + C$. Now we can factor the numerator: $\ln\left|\frac{(1-\cos\theta)(1+\cos\theta)}{1+\cos\theta}\right| + C$

This simplifies to: $\ln\left|1 - \cos\theta\right| + C$.

EXAMPLE: Find the indefinite integral: $\int (\sin 2\theta) e^{\cos^2\theta}\ d\theta$.

Using substitution we will get $u = \cos^2\theta$. Then $du = -2\cos\theta\sin\theta\ d\theta$. Solving for $d\theta$ we get $d\theta = \frac{du}{-2\cos\theta\sin\theta}$. We can use the trig identity: $\sin 2\theta = 2\sin\theta\cos\theta$. So $d\theta = \frac{du}{-\sin 2\theta}$. Now we make our substitution: $\int (\sin 2\theta)\cdot e^u \cdot \frac{du}{-\sin 2\theta}$. This simplifies to: $-\int e^u\ du$. When we integrate this we get $-e^u + C$. Then we can replace the u with $\cos^2\theta$ and we get: $-e^{\cos^2\theta} + C$.

Change of Variables

In all the problems we did we were able to always cancel out the x terms and just have u. In the next two problems this will not automatically happen so we need to change variables.

EXAMPLE: Integrate by substitution: $\int x(1-x)^4 \ dx$.

First we will let $u = 1-x$. Then $du = -dx$. Solving for dx we get $dx = -du$. Now we will substitute this for dx and we will substitute a u for $1-x$. Now we have $-\int x \cdot u^4 \ du$. The problem here is that all the x terms did not cancel automatically. What we can do not is to use our formula for u which is $u = 1-x$ and then solve for x. You will get $x = 1-u$. Now we can substitute $1-u$ for x giving you: $-\int (1-u)\cdot u^4 \ dx$. After multiplying we will get: $-\int u^4 - u^5 \ du$. The antiderivative is $-\frac{u^5}{5} + \frac{u^6}{6} + C$. Now substitute back in the x: $\int x(1-x)^4 \ dx = -\frac{(1-x)^5}{5} + \frac{(1-x)^6}{6} + C$.

EXAMPLE: Integrate by substitution: $\int \frac{2x+1}{\sqrt{x+4}} \ dx$.

First we will let $u = x+4$. Then $du = dx$. Now we will substitute this for dx and we will substitute a u for $x+4$. Now we have $\int \frac{2x+1}{\sqrt{u}} \ du$. The problem here is that all the x terms did not cancel automatically. What we can do not is to use our formula for u which is $u = x+4$ and then solve for x. You will get $x = u-4$. Now we can substitute $u-4$ for x giving you: $\int \frac{2(u-4)+1}{\sqrt{u}} \ du$. This simplifies to: $\int \frac{2u-7}{\sqrt{u}} \ du$. We can divide each term in the top by the bottom to get: $\int 2u^{\frac{1}{2}} - 7u^{-\frac{1}{2}} \ du$. The antiderivative is $2\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{7u^{\frac{1}{2}}}{\frac{1}{2}} + C$. This simplifies to: $\frac{4}{3}u^{\frac{3}{2}} - 14u^{\frac{1}{2}} + C$ Now we substitute back in the x: $\int \frac{2x+1}{\sqrt{x+4}} \ dx = \frac{4}{3}(x+4)^{\frac{3}{2}} - 14(x+4)^{\frac{1}{2}} + C$.