Section 1.3 Trigonometric Functions

Angles are measured a couple of different ways. The first unit of measurement is a **degree** in which 360° (degrees) is equal to one revolution.

Another unit of measurement for angles is **radians**. In radians, 2π is equal to one revolution. So a conversion between radians and degrees is $2\pi = 360^\circ$, or $\pi = 180^\circ$.

When converting from degrees to radians:

Multiply your degrees by $\frac{\pi}{180}$

When converting from radians to degrees:

Multiply your radians by $\frac{180}{\pi}$

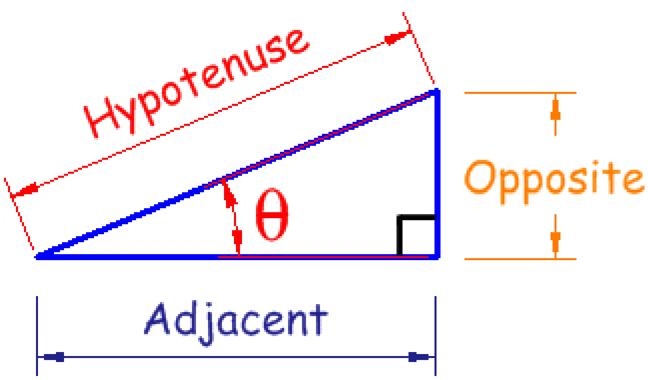
EXAMPLE: Convert 60° to radians.

We will take 60 and multiply it by $\frac{\pi}{180}$ and you will get: $60 \cdot \frac{\pi}{180}$. This reduces to $\frac{\pi}{3}$.

EXAMPLE: Convert $\frac{4\pi}{3}$ into degrees.

We will take $\frac{4\pi}{3}$ and multiply it by $\frac{180}{\pi}$ and you will get: $\frac{4\pi}{3} \cdot \frac{180}{\pi}$. This reduces to 240°.

Right Triangle Trig Definitions



$$\sin\theta = \frac{opposite}{hypotenuse} \qquad \csc\theta = \frac{hypotenuse}{opposite}$$

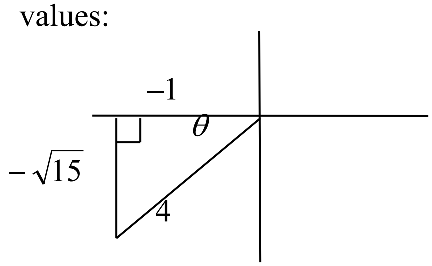
$$\cos\theta = \frac{adjacent}{hypotenuse} \qquad \sec\theta = \frac{hypotenuse}{adjacent}$$

$$\tan\theta = \frac{opposite}{adjacent} \qquad \cot\theta = \frac{adjacent}{opposite}$$

EXAMPLE: Given $\cos\theta = -\frac{1}{4}$ and $\sin\theta < 0$, find the exact value of the six trig functions.

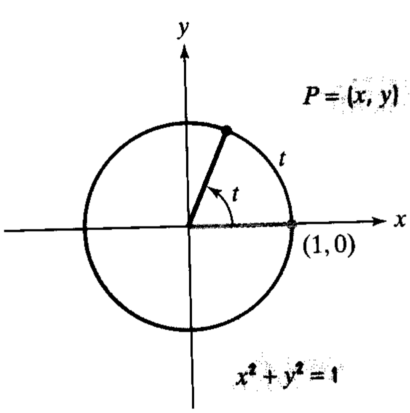
First we need to draw the triangle like we did in a previous section. This time we are told $\sin\theta < 0$, which means we need to draw the triangle in the quadrant in which both cosine and sine are negative, which would be the third quadrant. Our fraction is negative. That means that either 1 or 4 must be negative when we put this in our drawing. The hypotenuse is NEVER negative, so this means that 1 must be negative since this is the adjacent side. Our θ is drawn at the origin, and this is always where it will be drawn. This is like a reference angle.

We can use the Pythagorean theorem to find the missing side: $a^2 + (-1)^2 = 4^2$. Solving this you will get $a = \pm\sqrt{15}$. In our drawing, since we are in the third quadrant, we MUST use the negative answer. The reason why is this vertical distance is really a y value, and if we think about it in terms of graphing something, the y would be negative since we are below the x-axis. So now our drawing is complete and we can find the six trig values:



$\sin\theta = -\frac{\sqrt{15}}{4}$, $\csc\theta = -\frac{4}{\sqrt{15}} = -\frac{4\sqrt{15}}{15}$, $\cos\theta = -\frac{1}{4}$, $\sec\theta = -4$, $\tan\theta = \sqrt{15}$, $\cot\theta = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15}$

A **unit circle** is a circle centered at the origin with a radius of 1. It is shown in the drawing below. Here the letter t represents an angle measure. The point P = (x, y) represents a point on the unit circle.



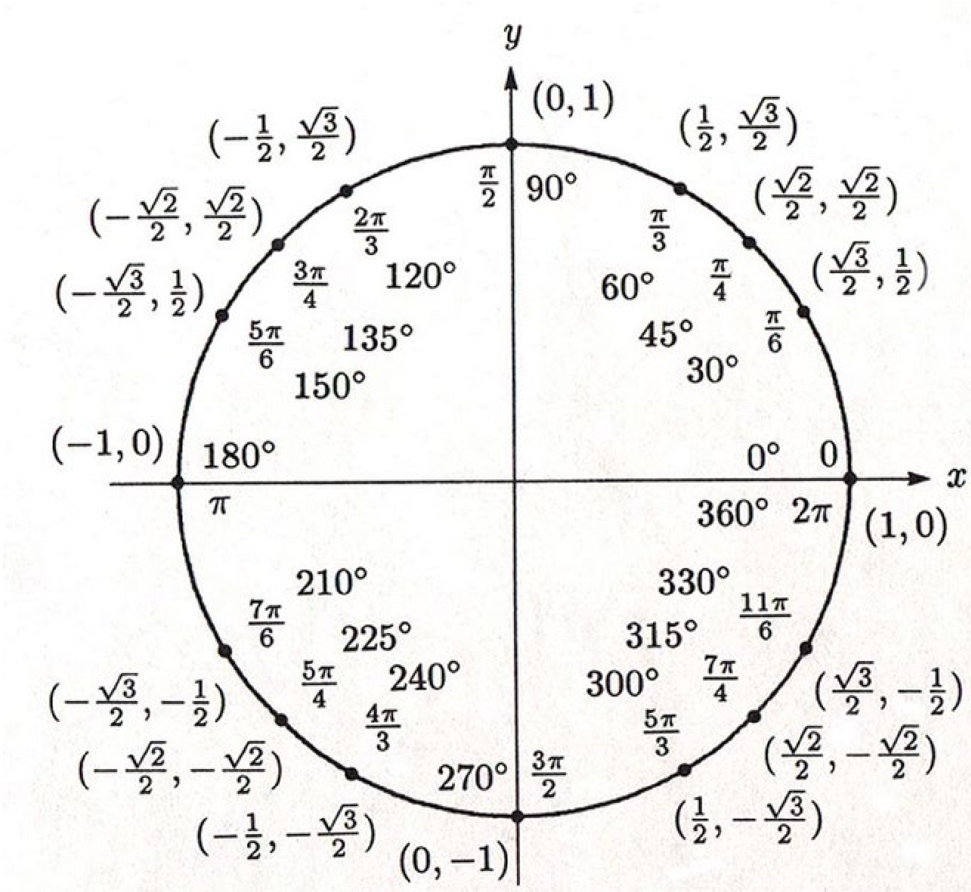
The following definitions are given based on this picture.

$\cos(t) = x$ $\quad \sec(t) = \frac{1}{x}$

$\sin(t) = y$ $\quad \csc(t) = \frac{1}{y}$

$\tan(t) = \frac{y}{x}$ $\quad \cot(t) = \frac{x}{y}$

The angle t can be any angle between 0 and 360 degrees. There are certain places on the unit circle in which we have exact values. The values are given below. This unit circle contains both degrees and radians.



EXAMPLE: Find the exact value of $\cos 135^\circ$ without using the cosine function on a calculator.

To find the value, we can use the unit circle. We look at the angle of 135 degrees, and indicate the *x*-value since the *x*-value represents cosine. Therefore, $\cos 135^\circ = -\frac{\sqrt{2}}{2}$.

EXAMPLE: Find the exact value of $\tan\left(-\frac{2\pi}{3}\right)$ without using the tangent function on a calculator.

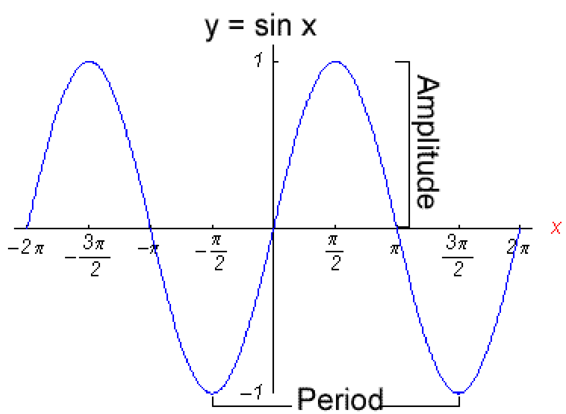
We want to make this a positive angle. We can do this be adding a revolution. Adding one revolution will result in the same location on the unit circle. One revolution is 2π radians. So $-\frac{2\pi}{3} + 2\pi = \frac{4\pi}{3}$.

So $\tan\left(-\frac{2\pi}{3}\right) = \tan\left(\frac{4\pi}{3}\right)$. We look on the unit circle at $\frac{4\pi}{3}$ radians. The tangent will be the *y*-value divided by the *x*-value. Therefore, $\tan\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2} \Big/ -\frac{1}{2} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \sqrt{3}$.

EXAMPLE: Find the exact value of $\sin\left(-\frac{11\pi}{3}\right)$ without using sine function on a calculator.

Instead of working in radians, we can change our angle into degrees: $-\frac{11\pi}{3}\cdot\frac{180}{\pi}=-660^\circ$. So now our problem becomes: $\sin(-660^\circ)$. Next, we want to make this a positive angle. We can do this be adding two revolutions, since just adding one revolution will still result in a negative angle. So $-660^\circ+360^\circ+360^\circ=60^\circ$. So now our problem becomes $\sin(60^\circ)$. We look at the angle of 60 degrees, and indicate the *y*-value since the *y*-value represents sine, and we will get $\frac{\sqrt{3}}{2}$. Therefore, $\sin\left(-\frac{11\pi}{3}\right)=\frac{\sqrt{3}}{2}$.

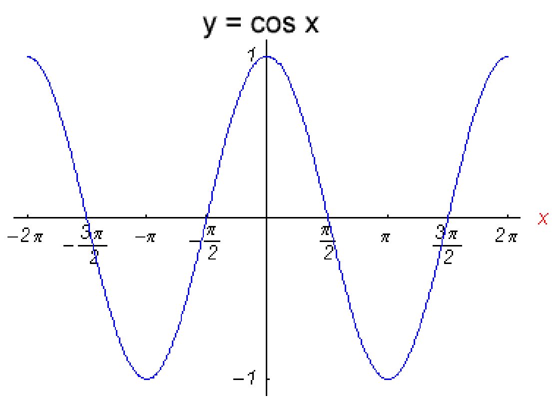
Graphs of Sine and Cosine



Period: How long it takes the graph to repeat itself
For sine graphs, the period is 2π .

Amplitude $=\dfrac{Highest\ \ value - Lowest\ \ value}{2}$

For the regular sine graph the amplitude is 1.



The period for cosine graphs is 2π

The amplitude for a regular cosine graph is 1.

General Form of a Sine or Cosine Equation:

$y = A\sin(Bx - C)$ or $y = A\cos(Bx - C)$

Amplitude $=|A|$, **Period** $=\frac{2\pi}{B}$, **Phase Shift** $=\frac{opp\ \ sign\ \ of\ \ C}{B}$

The **phase shift** is a shift of the graph to the left or to the right.

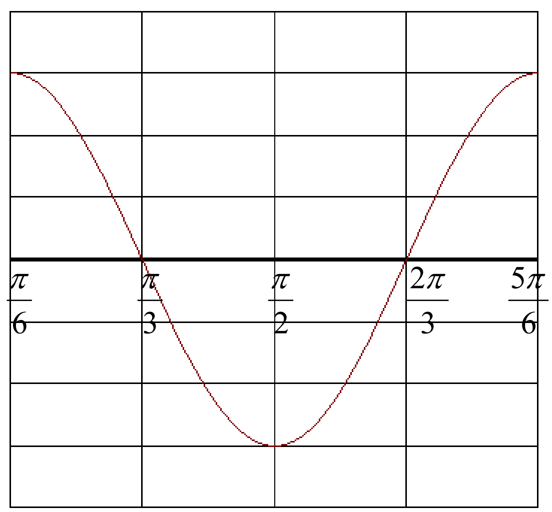
EXAMPLE: Identify the amplitude, period, phase shift and graph of $y = 3\cos\left(3x - \frac{\pi}{2}\right)$. (Graph 1 period).

First the amplitude is $|3| = 3$. The period is $\frac{2\pi}{3}$. To find the phase shift, we take the opposite sign of C and divide it by B. Then the phase shift is $\frac{\frac{\pi}{2}}{3} = \frac{\pi}{6}$. This tells us the graph starts at $\frac{\pi}{6}$ because this is the phase shift.

We need to find our 5 key points by finding the quarter point. In this problem, the quarter point is $\frac{\frac{2\pi}{3}}{4} = \frac{\pi}{6}$.
We will start with the left key point $\frac{\pi}{6}$ and we will keep adding our quarter point to this to generate the other key points:

We start with $\frac{\pi}{6}$. Then we have: $\frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6}$, $\frac{2\pi}{6} + \frac{\pi}{6} = \frac{3\pi}{6}$, $\frac{3\pi}{6} + \frac{\pi}{6} = \frac{4\pi}{6}$, $\frac{4\pi}{6} + \frac{\pi}{6} = \frac{5\pi}{6}$.

Now you can reduce each of your key points to the following: $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}$ and then graph:



Remember the cosine graph always starts at the amplitude, which is 3 in this case.

List of Trigonometric Identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$ $\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

$\sin^2\theta + \cos^2\theta = 1$ $\sin^2\theta = 1 - \cos^2\theta$ $\cos^2\theta = 1 - \sin^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$ $\tan^2\theta = \sec^2\theta - 1$ $\csc^2\theta = 1 + \cot^2\theta$ $\cot^2\theta = \csc^2\theta - 1$

Proving a Trigonometric Identity

Although we will not specifically be doing these types of problems in Calculus, these problems are good review of your trig identities.

EXAMPLE: Establish the identity: $\dfrac{\cos x}{1+\sin x}+\dfrac{1+\sin x}{\cos x}=2\sec x$.

You want to show that one side of the equation equals the other side. In these problems you are NOT allowed to do operations like adding or subtracting things from one side to the other. Think of each side as independent.

Once again we want to first get a single fraction so we need common denominators.

$\dfrac{\cos x}{1+\sin x}\cdot\left(\dfrac{\cos x}{\cos x}\right)+\dfrac{1+\sin x}{\cos x}\cdot\left(\dfrac{1+\sin x}{1+\sin x}\right)=2\sec x$ Now multiply and write as a single fraction.

$\dfrac{\cos^2 x+(1+\sin x)^2}{\cos x(1+\sin x)}=2\sec x$ We will expand the numerator.

$\dfrac{\cos^2 x+\sin^2 x+2\sin x+1}{\cos x(1+\sin x)}=2\sec x$ We will use the identity $\cos^2 x+\sin^2 x=1$

$\dfrac{1+2\sin x+1}{\cos x(1+\sin x)}=2\sec x$ Simplify the numerator.

$\dfrac{2\sin x+2}{\cos x(1+\sin x)}=2\sec x$ Factor the numerator.

$\dfrac{2(\sin x+1)}{\cos x(1+\sin x)}=2\sec x$ We can cancel the $\sin x+1$ from the top and bottom.

$\dfrac{2}{\cos x}=2\sec x$ We will use the identity $\sec x=\dfrac{1}{\cos x}$.

$2\sec x=2\sec x$ Both sides are the same, so we are done.

Solving Trigonometric Equations

EXAMPLE: Solve the equation: $2\cos^2 x + \cos x - 1 = 0$ on $[0,\ 2\pi)$.

We can factor this one: $(2\cos x - 1)(\cos x + 1) = 0$. Now set each part equal to zero. We get $2\cos x - 1 = 0$ and $\cos x + 1 = 0$. Solving the first equation we will get $\cos x = \frac{1}{2}$. We look on the unit circle and look for any x values that are $\frac{1}{2}$. This will happen at $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. Solving the second equation you will get $\cos x = -1$. The angle that gives an x value of negative one is π. Therefore our answers are $\frac{\pi}{3}$, π, and $\frac{5\pi}{3}$.

EXAMPLE: Solve the equation: $\sin x \cos^2 x = 2\sin x$ on $[0,\ 2\pi)$.

We need to set this equal to zero: $\sin x \cos^2 x - 2\sin x = 0$. Now factor out a common factor of $\sin x$: $\sin x(\cos^2 x - 2) = 0$. Now set each factor equal to zero. We have $\sin x = 0$. Looking at our unit circle we see that x is 0 and π. The other equation gives us $\cos^2 x = 2$. Taking the square root we get $\cos x = \pm\sqrt{2}$. This means that $\cos x = \pm 1.41$. Since this number is larger than one, this will not give us any solutions because of the domain of the cosine. So our answers for x are 0 and π.

EXAMPLE: Solve the equation: $2\sin^2\theta - \cos 2\theta = 0$ on $[0,\ 2\pi)$.

Since there are both sines and cosines we need to use an identity to get all the terms to have the same trig value. Since I notice there is already a sine in the problem I want to use $\cos 2\theta = 1 - 2\sin^2\theta$. So now the problem is: $2\sin^2\theta - (1 - 2\sin^2\theta) = 0$. Simplifying we get: $4\sin^2\theta - 1 = 0$. When we solve this we get $\sin^2\theta = \pm\frac{1}{4}$ so $\sin\theta = \pm\frac{1}{2}$. Then values off the unit circle are: $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

EXAMPLE: Solve the equation: $2\sin^2\theta - \cos 2\theta = 0$ on $[0,\ 360^\circ)$.

Since there are both sines and cosines we need to use an identity to get all the terms to have the same trig value. Since I notice there is already a sine in the problem I want to use $\cos 2\theta = 1 - 2\sin^2\theta$. So now the problem is: $2\sin^2\theta - (1 - 2\sin^2\theta) = 0$. Simplifying we get: $4\sin^2\theta - 1 = 0$. When we solve this we get $\sin^2\theta = \pm\frac{1}{4}$ so $\sin\theta = \pm\frac{1}{2}$. Then values off the unit circle are: 30°, 150°, 210°, 330°. Notice that since our interval was given in degrees we can write our answers in degrees.