1.4 Inverse Functions

One-To-One Function

A function f is a one-to-one function if for a and b in the domain of f, if $a \neq b$, then $f(a) \neq f(b)$, or equivalently, if $f(a) = f(b)$ then $a = b$. In other words, for each y value there can only be one x value.

EXAMPLE: For each function below, determine whether each is one-to-one.

{(1, 2), (3, 7), (2, 9), (8, 11)}

This function is one-to-one because each *y* value has only one *x* value.

{(-3, 4), (5, 6), (7, 4), (-2, 3)}

This function is not one-to-one because the x-values -3 and 7 both go to 4. In order to be one-to-one, each y value must go to only one x value.

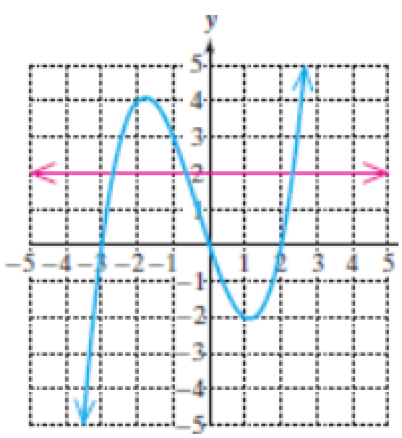
{(-2, 4), (-1, 6), (0, 3), (-2, 8)}

<u>NOT</u> a function because when x is -2 it goes to both 4 and 8. There are two different y values for one x. In order to be one-to-one it first must be a function. So since this is not a function it cannot be one-to-one.

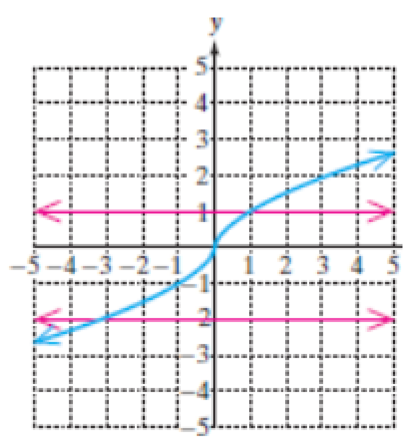
Horizontal Line Test

If you pass a horizontal line and it hits the graph at only one place, then it is one-to-one.
EXAMPLE: Use the horizontal line test to determine if the graph below defines y as a one-to-one function.

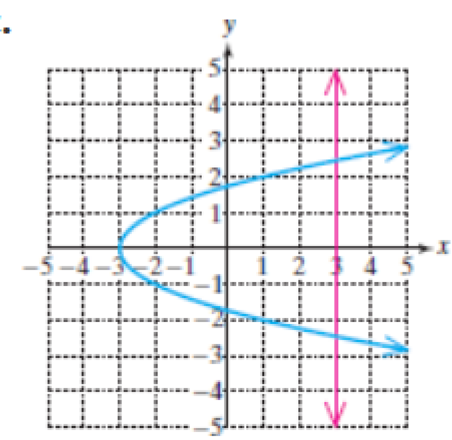
a.



b.



c.



For part **a**, the horizontal line hits the graph three times. Therefore it is not one-to-one.
For part **b**, the horizontal line hits the graph only once no matter where it is drawn. Therefore it is one-to-one.
For part **c**, it does not pass the vertical line test. Therefore it is not a function, and not one-to-one.

Inverse Function:

Notation to write "the inverse of $f(x)$" is $f^{-1}(x)$. If two functions *f* and f^{-1} are inverses then the following must be true: $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$. Note that these functions must be one-to-one.

EXAMPLE: Given $f(x) = 2x - 1$ and $f^{-1}(x) = \frac{1}{2}x + \frac{1}{2}$ verify that they are inverses.

We need to show $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$. Let's first find $(f \circ f^{-1})(x)$.

$(f \circ f^{-1})(x) = f(f^{-1}(x))$ Start with the definition.

$(f \circ f^{-1})(x) = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ We remove the $f^{-1}(x)$ and replace it with $\frac{1}{2}x + \frac{1}{2}$.

$(f \circ f^{-1})(x) = 2\left(\frac{1}{2}x + \frac{1}{2}\right) - 1$ Now simplify.

$(f \circ f^{-1})(x) = x + 1 - 1$

$(f \circ f^{-1})(x) = x$ We have shown this is true. Now we need to show $(f^{-1} \circ f)(x) = x$.

$(f^{-1} \circ f)(x) = f^{-1}(f(x))$ First start with the definition.

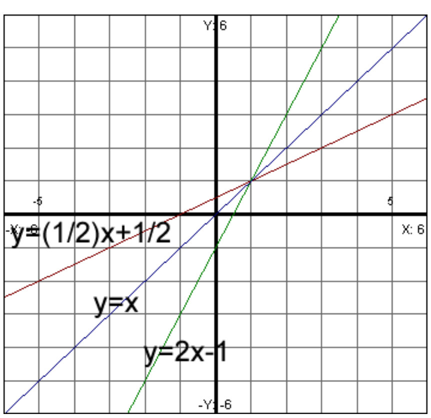
$(f^{-1} \circ f)(x) = f^{-1}(2x - 1)$ We remove the f(x) and replace it with 2x – 1.

$(f^{-1} \circ f)(x) = \frac{1}{2}(2x - 1) + \frac{1}{2}$ Now simplify.

$(f^{-1} \circ f)(x) = x - \frac{1}{2} + \frac{1}{2}$

$(f^{-1} \circ f)(x) = x$ We have shown this is true. So we have verified they are inverses.

What is the significance of the *x*? Why do we get *x* when we simplify? I'm glad you asked! Let's look at the graph of *f* and f^{-1}. I will also graph $y = x$. Notice that *f* and f^{-1} are symmetric to the line $y = x$. This is always the case with inverses. Notice also that points on the graph of *f*(*x*) are reversed on $f^{-1}(x)$. For example, on the *f*(*x*) line we see the points (2, 3) and (-1, -3). On the graph of $f^{-1}(x)$ we get the points (3, 2) and (-3, -1).



How to find an inverse algebraically:

Step 1: Replace $f(x)$ with *y*.
Step 2: Switch *x* and *y*.
Step 3: Solve for *y*.
Step 4: Replace *y* with $f^{-1}(x)$.

EXAMPLE: Given $f(x)=\sqrt{x+7}$ find $f^{-1}(x)$.
Step 1: $y=\sqrt{x+7}$
Step 2: $x=\sqrt{y+7}$
Step 3: $(x)^2=\left(\sqrt{y+7}\right)^2$
$x^2=y+7$
$x^2-7=y$
Step 4: $x^2-7=f^{-1}(x)$

Note that the domain of $f(x)$ is $[-7,\infty)$ and the range of is $f(x)$ is $[0,\infty)$. If you switch these you will get the domain and range for the inverse. The domain of $f^{-1}(x)$ is $[0,\infty)$ and the range of is $f^{-1}(x)$ is $[-7,\infty)$. So we need to restrict the domain on our answer. $f^{-1}(x)=x^2-7$ where $x>0$.

Inverse Trigonometric Functions

From trigonometry we know that $\sin 30^\circ=\frac{1}{2}$. We put in an angle and get a value as a result. In inverse trig functions we put in the value and get an angle: $\sin^{-1}\frac{1}{2}=30^\circ$. So here we put in the value of one half and got 30 degrees as a result. We are not allowed to put any number into our inverse trig functions. There are restrictions on the domain that are given in the following table:

	Domain	Range
$y=\sin^{-1}x$	$-1\le x\le 1$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}$
$y=\cos^{-1}x$	$-1\le x\le 1$	$0\le y\le\pi$
$y=\tan^{-1}x$	$-\infty<x<\infty$	$-\frac{\pi}{2}<y<\frac{\pi}{2}$

We are not going to look at the graphs of the inverse trig functions in this class, however that is where the domain and range is derived from.

EXAMPLE: Find $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

What this is really asking is: "find an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ that has a value of $\frac{\sqrt{3}}{2}$." You will need to remember your unit circle or table for this one. This corresponds to an angle of 60 degrees, or $\frac{\pi}{3}$, which is the answer.

EXAMPLE: Find $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

What this is really asking is: "find an angle between 0 and π that has a value of $\frac{1}{\sqrt{2}}$." If you rationalize, then $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. This corresponds to an angle of 45 degrees, or $\frac{\pi}{4}$, which is the answer.

EXAMPLE: Find $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$.

What this is really asking is: "find an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ that has a value of $\frac{\sqrt{3}}{3}$." You will need to remember your unit circle or table for this one. This corresponds to an angle of 30 degrees.

EXAMPLE: Find $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$.

From our earlier example, we know that $\sin^{-1}\frac{1}{2} = 30^\circ$. So then our problem becomes $\cos\left(30^\circ\right)$. From the unit circle, $\cos 30^\circ = \frac{\sqrt{3}}{2}$. Therefore, $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = \cos\left(30^\circ\right) = \frac{\sqrt{3}}{2}$.

EXAMPLE: Find the exact value: $\cos^{-1}\left(\cos\frac{\pi}{3}\right)$.

The unit circle tells us that $\cos\frac{\pi}{3} = \frac{1}{2}$. So our problem becomes $\cos^{-1}\left(\frac{1}{2}\right)$. We can look at the unit circle for this value, and we get $\frac{\pi}{3}$. There is also a property $\cos^{-1}\left(\cos x\right) = x$ as long as $0 \le x \le \pi$, so either of these processes will allow us to find the answer,. Therefore, $\cos^{-1}\left(\cos\frac{\pi}{3}\right) = \frac{\pi}{3}$.