1.5 Exponential and Logarithmic Functions

The first part of this section will be a review over exponents. Listed below are rules for exponents.

Exponent Laws:

$a^s \cdot a^t = a^{s+t}$ Example: $2^5 \cdot 2^3 = 2^{5+3} = 2^8$

$\frac{a^s}{a^t} = a^{s-t}$ Example: $\frac{2^6}{2^3} = 2^{6-3} = 2^3$

$(a^s)^t = a^{s \cdot t}$ Example: $(2^3)^5 = 2^{3 \cdot 5} = 2^{15}$

$a^{-s} = \frac{1}{a^s}$ Example: $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

$\left(\frac{a}{b}\right)^{-s} = \left(\frac{b}{a}\right)^s$ Example: $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

EXAMPLE: Use laws of exponents to simplify: $\left(17^{\sqrt{2}}\right)^{\frac{\sqrt{2}}{2}}$.

In this case we will use rule #4. We have a power raised to another power: $17^{\frac{\sqrt{2}}{1} \cdot \frac{\sqrt{2}}{2}} = 17^{\frac{2}{2}} = 17$

EXAMPLE: Use laws of exponents to simplify: $\left(4^7 \cdot 4^3\right)^{\frac{1}{5}}$.

First we will use rule #1 to simplify inside: $\left(4^{7+3}\right)^{\frac{1}{5}} = \left(4^{10}\right)^{\frac{1}{5}}$. Now we use rule #4: $4^{\frac{10}{1} \cdot \frac{1}{5}} = 4^2 = 16$

EXAMPLE: Simplify: $\left(\frac{3x^4 y^{-2}}{x^{-3} y}\right)^{-4} \left(\frac{y}{x}\right)^{-2}$ and write with positive exponents.

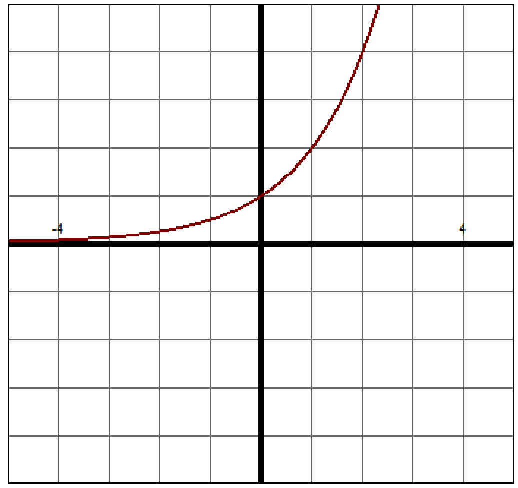
First I will flip the fraction to make the outside number positive: $\left(\frac{x^{-3} y}{3x^4 y^{-2}}\right)^4 \left(\frac{x}{y}\right)^2$. Now raise everything to the power of 4. In doing so, we multiply the exponents: $\frac{x^{-12} y^4}{81x^{16} y^{-8}} \cdot \frac{x^2}{y^2}$. We will multiply across the top and bottom. We will need to add exponents: $\frac{x^{-10} y^4}{81x^{16} y^{-6}}$. Now subtract exponents: $\frac{1}{81} x^{-26} y^{10}$. Finally we can write with positive exponents: $\frac{y^{10}}{81x^{26}}$.

Exponential function: $y = b^x$

We will look at a specific exponential function to see its characteristics. To do this we will make a table. Then we will plot the points. The graph will be a curved line:

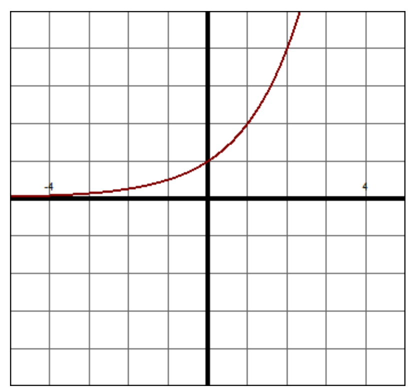
Graph of $y = 2^x$

x	$y = 2^x$	(x, y)
-2	$y = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$y = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$y = 2^0 = 1$	$(0, 1)$
1	$y = 2^1 = 2$	$(1, 2)$
2	$y = 2^2 = 4$	$(2, 4)$

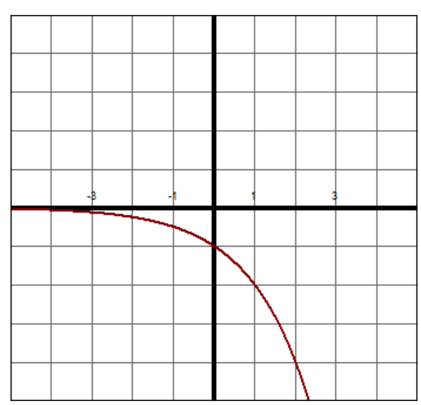


Notice from the graph of $y = 2^x$ that the y-intercept is (0, 1). This will always be the case for exponential functions. Also notice that there is a horizontal asymptote at y = 0.

EXAMPLE: Graph using transformations: $y = -2^x$.
Indicate the domain and range. State the horizontal asymptote.



We start with the base graph $y = 2^x$. The negative this time is in front of the number being raised to the power of x. This is another transformation. When this happens it will flip the graph over the horizontal axis. Notice that the original y-intercept is also reflected over the horizontal axis. It was originally at (0, 1) but now it is at (0, –1).



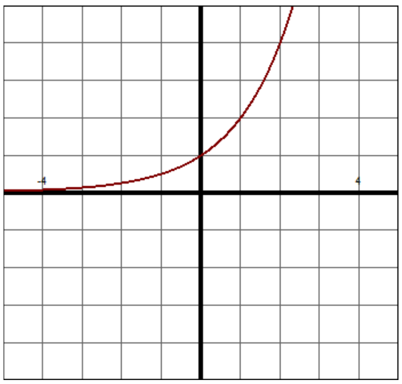
The horizontal asymptote did not shift, so it is the same as our base graph, which is y = 0.

No restrictions on x-values, so the domain is $(-\infty, \infty)$.
We see that the graph is only using negative values, so the range is $(-\infty, 0)$.

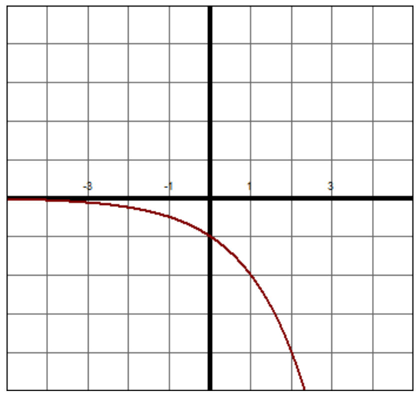
EXAMPLE: Graph using transformations: $y = -\left(\frac{1}{2}\right)^x + 3$.

Indicate the domain and range. State the horizontal asymptote.

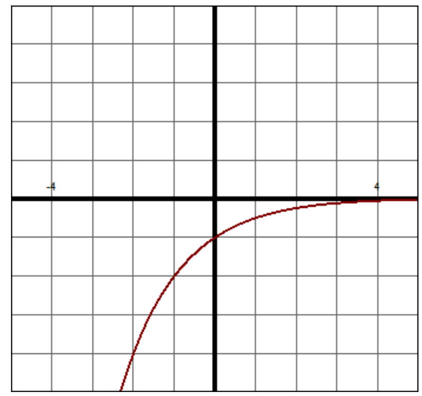
First we want to rewrite the original as $y = -\left(2^{-1}\right)^x + 3$. Then we write $y = -2^{-x} + 3$



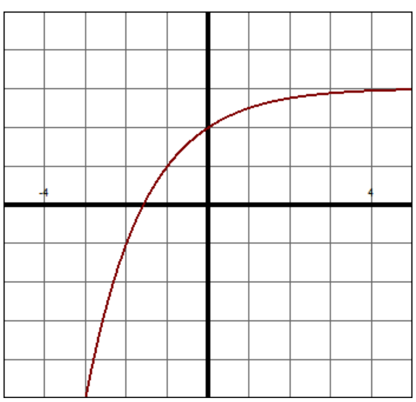
Start with our base graph $y = 2^x$
Our y-intercept is at (0, 1).



Next we graph $y = -2^x$ which flips the graph $y = 2^x$ over the horizontal axis. Notice the point (0, 1) got flipped over the horizontal axis, so the new point is now at (0, –1).



To graph $y = -2^{-x}$ we will flip $y = -2^x$ over the vertical axis. Notice our point at (0, –1) is still at the same location.



Now we move the graph of $y = -2^{-x}$ up three units. The point (0, –1) has move up three units to (0, 2). This graph is our final answer.

In our final answer, the horizontal asymptote is y = 3. The domain is $(-\infty, \infty)$ and range $(-\infty, 3)$.

Logarithms

In the last section we looked at inverses. In order to find an inverse we need to switch *x* and *y*.

Suppose we wanted to find the inverse of our exponent function, $y = b^x$. First we need to switch *x* and *y*. We will get $x = b^y$. How do we solve for *y*? This is where we need logarithms, which are a way to solve for an exponent.

With logarithms there are two forms: **Exponential form**: $x = b^y$ **Logarithmic form**: $y = \log_b x$

EXAMPLE: Change $\log_c 6 = 8$ into exponential form.

Here b = c, y = 8 and x = 6. If we put these into the exponential form we get $c^8 = 6$.

EXAMPLE: Change $2^d = 8$ into logarithmic form.

Here x = 8, b = 2 and y = d. If we put these into the logarithmic form we get $d = \log_2 8$.

Equal Bases Property

If $a^u = a^v$ then u = v.

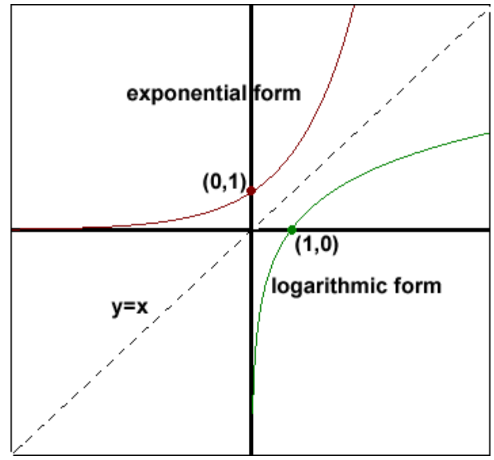
EXAMPLE: Find the exact value of $\log_4 64$.

This one we can't do on our calculator since we don't have a base 4. What we can do is change it into exponential form and solve. First we will let $y = \log_4 64$. Changing into exponential form we will get $4^y = 64$. We want to use the Equal Bases Property. To do this, we need to rewrite 64 so that it has a base of 4. This is 4^3, since (4)(4)(4) = 64. So since $4^y = 4^3$ we can set the exponents equal, resulting in *y* = 3. So now we know that $\log_4 64 = 3$.

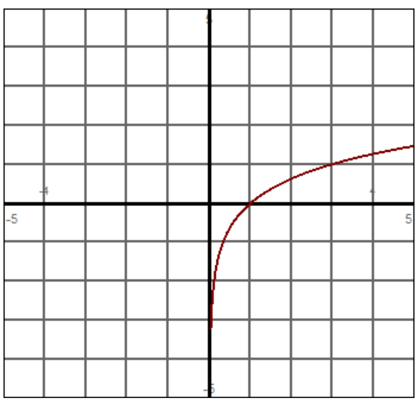
EXAMPLE: Find the exact value of $\log \frac{1}{10000}$.

Remember that this is the same as . $\log_{10} \frac{1}{10000}$. Now we will change it into exponential form and solve. First we will let $y = \log_{10} \frac{1}{10000}$. Changing into exponential form we will get $10^y = \frac{1}{10000}$. We want to use the Equal Bases Property. To do this, we need to rewrite the right hand of the equation so that it has a base of 10. So 10000 = 10^4. But since it is on the bottom of a fraction then $\frac{1}{10000} = 10^{-4}$. So since $10^y = 10^{-4}$ we can set the exponents equal, resulting in *y* = –4. So now we know that $\log \frac{1}{10000} = -4$.

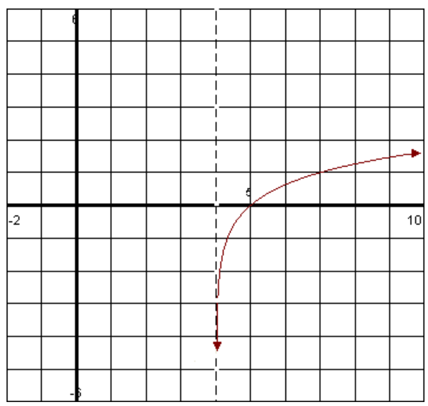
Let's try and draw a graph of $y = \log_b x$. In order to do this, we need to first draw the graph of $x = b^y$. Then we need to draw its inverse. We know that the log function is the inverse of the exponential function. We also know that all inverse are reflective about the line y = x. In the graph to the right you can see both graphs drawn. We see that the point (0, 1) and (1, 0) are flipped, which it should since they are inverses. So the graph of $y = \log_b x$ will always cross over the x-axis one unit away from the vertical asymptote. Let's look at the graph of the logarithmic form. The y-axis is a vertical asymptote. There is no horizontal asymptote on the logarithmic graph. From the graph we can conclude:



EXAMPLE: Graph using transformations: $y = \log_3(x-4)$.

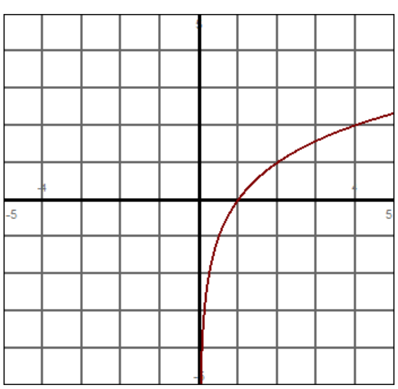


Start with our base graph $y = \log_b x$

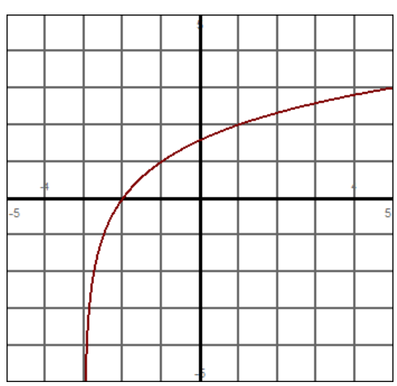


Because of transformation rules, we move $y = \log_b x$ four places to the right. Note the dotted line indicating the vertical asymptote.

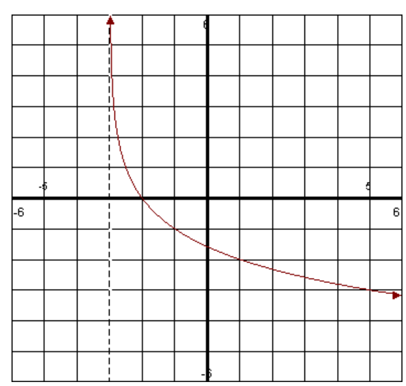
EXAMPLE: Graph using transformations: $y = -\log_2(x+3)$.



Start with our base graph $y = \log_b x$



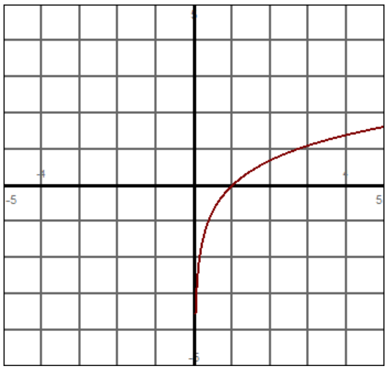
We move $y = \log_b x$ three places to the left. This is the graph of $y = \log_2(x+3)$.



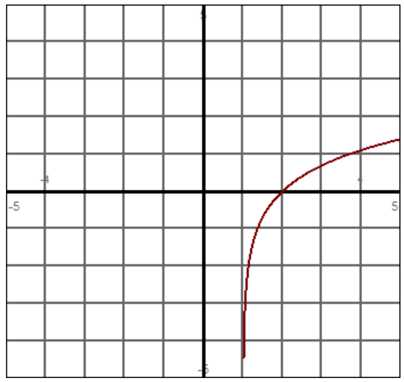
Because of the negative in front of the log, we flip the graph over the horizontal axis.

EXAMPLE: Graph using transformations: $y = \ln(1 - x)$.

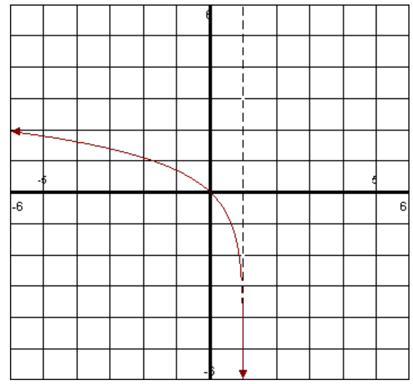
This is equivalent to $y = \log_e(1 - x)$ which will have the same initial shape as the previous two examples. Before we can use transformations we first need to get the x to come first. We will get $y = \log_e(-x + 1)$. Now we factor out a -1: $y = \log_e\left(-(x - 1)\right)$. So we will now graph $y = \ln\left(-(x - 1)\right)$ using transformations.



Start with our base graph $y = \ln x$



We move $y = \ln x$ one place to the right. This is the graph of $y = \ln(x - 1)$.



Because of the negative inside of the log, we flip the graph over the vertical axis. We indicate the vertical asymptote with a dotted line.

Algebraic Properties of Logarithms

For any numbers $x > 0$ and $y > 0$,

1.) *Product Rule*: $\log_a xy = \log_a x + \log_a y$

2.) *Quotient Rule*: $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

3.) *Reciprocal Rule*: $\log_a\left(\frac{1}{y}\right) = -\log_a y$

4.) *Power Rule*: $\log_a x^y = y \log_a x$

Inverse Properties for a^x and $\log_a x$

1.) Base a: $a^{\log_a x} = x$, $\log_a a^x = x$ $\quad a > 0,\ a \neq 1,\ x > 0$

2.) Base e: $e^{\ln x} = x$, $\ln e^x = x$ $\quad x > 0$

EXAMPLE: Find the exact value using logarithm properties: $\log_3\left(\frac{1}{3}\right)$.

First use log property #3: $-\log_3 3$. Since $\log_3 3 = 1$, our answer is -1.

EXAMPLE: Find the exact value using logarithm properties: $\log_{144} 12$.

First we will rewrite this one so that the number after the log is the same as the base: $\log_{144} \sqrt{144}$. This can be written as $\log_{144} 144^{\frac{1}{2}}$. Then we will use log property #4: $\frac{1}{2}\log_{144} 144$. Since $\log_{144} 144 = 1$, our answer is $\frac{1}{2}$.

EXAMPLE: Find the exact value using logarithm properties: $e^{\ln 6 - \ln 7}$.

First we will use log property #2. This will give us: $e^{\ln\left(\frac{6}{7}\right)}$. Then use inverse property #1, so the answer is $\frac{6}{7}$.

EXAMPLE: Express $\log_9 x^2 \cdot \sqrt{3x-5}$ as a sum or difference of logarithms. Express powers as factors.

Apply property #1. You will get $\log_9 x^2 + \log_9 \sqrt{3x-5}$. The square root can be written as a ½ power: $\log_9 x^2 + \log_9 (3x-5)^{\frac{1}{2}}$. Now use property #5 to bring the powers down in front of the logs since it wants us to express powers as factors: $2 \cdot \log_9 x + \frac{1}{2} \cdot \log_9 (3x-5)$. This is our answer.

EXAMPLE: Express $\ln \frac{(x+5)^4}{x^3}$ as a sum or difference of logarithms. Express powers as factors.

Apply property #2. You will get $\ln(x+5)^4 - \ln x^3$. Now use property #5 to bring the powers down in front of the logs since it wants us to express powers as factors: $4 \cdot \ln(x+5) - 3\ln x$.

EXAMPLE: Express $\log_4 \frac{(x-5)^5 \cdot \sqrt[3]{x-2}}{(x-1)^4}$ as a sum or difference of logarithms. Express powers as factors.

For this one you will apply both property #1 and #2. You will get $\log_4 (x-5)^5 \cdot \sqrt[3]{x-2} - \log_4 (x-1)^4$. Now we can use property #6 to break up the first log. You will get: $\log_4 (x-5)^5 + \log_4 \sqrt[3]{x-2} - \log_4 (x-1)^4$. We can rewrite the cube root as a 1/3 power: $\log_4 (x-5)^5 + \log_4 (x-2)^{\frac{1}{3}} - \log_4 (x-1)^4$. Now use property #5 to bring the powers down in front of the logs since it wants us to express powers as factors:
$5\log_4 (x-5) + \frac{1}{3}\log_4 (x-2) - 4 \cdot \log_4 (x-1)$.

Solving Logarithmic and Exponential Equations

EXAMPLE: Solve: $4^{x-2} - 64 = 0$.

First we isolate the exponential term: $4^{x-2} = 64$. In order to solve this, we must make both the bases the same. Since there is a 4 on the left hand side, I want to write 64 as 4 raised to some power. It is known that $4^3 = 64$ so we can now rewrite our equation:

$4^{x-2} = 4^3$. The Equivalence Property of Exponential Expressions states that if the bases are the same then we can set the exponents equal to each other. If we do this we will have x – 2 = 3. Solving this we get x = 5.

EXAMPLE: Solve: $3^x = 7$.

For this problem, we can't use the Equivalence Property of Exponential Expressions since one base can't be written in terms of the other. To solve this we will take either the natural log or regular log of both sides:

$\ln 3^x = \ln 7$	Now use property #5 to bring down the x.
$x \ln 3 = \ln 7$	Divide by sides by $\ln 3$ to solve for x.
$x = \dfrac{\ln 7}{\ln 3}$.	Note, you answer could also have been $x = \dfrac{\log 7}{\log 3}$, which is the same answer.

EXAMPLE: Solve: $e^{x+5} = 4$.

You definitely want to take the natural log of both sides so we can cancel out the e.

$\ln e^{x+5} = \ln 4$	This is the same as $\log_e e^{x+5} = \ln 4$. We can use property #4 to simplify this:
$x + 5 = \ln 4$	The ln and e cancel, so now we can solve for x.
$x = \ln 4 - 5$	This is our final answer.

If you are wondering if we can do subtract the 5 from 4 then the answer is no. These are not like terms.

EXAMPLE: Solve: $\log_5 (4x + 5) = 2$.

To solve this one, we first want to change from logarithmic form to exponential form. You will get $5^2 = 4x + 5$. So we have $25 = 4x + 5$, in which 20 = 4x, so x = 5.

EXAMPLE: Solve: $\log_2(x+11)+\log_2(x+7)=5$

$\log_2(x+11)(x+7)=5$	First combine into one log.
$2^5=(x+11)(x+7)$	Change into exponential form
$32=x^2+18x+77$	Multiply and simplify
$0=x^2+18x+45$	Set it equal to zero
$0=(x+3)(x+15)$	Factor
$x=-3,\ \ x=-15$	These are our answers. Now we need to make sure they are in domain.

If we put -3 into the original we get $\log_2(-3+11)+\log_2(-3+7)=5$ which is $\log_2 8+\log_2 4=5$ This is okay since both 8 and 4 are in the domain. If we put in -15 we get $\log_2(-15+11)+\log_2(-15+7)=5$ which results in $\log_2(-4)+\log_2(-8)=5$. We can't have negative numbers inside a log, therefore -15 is not one of our answers. Our only answer for this problem is $x=-3$.

EXAMPLE: Solve: $\log_2(x+3)-\log_2(x+5)=1$

$\log_2(x+3)-\log_2(x+5)=1$	First combine into one log. This time we will turn it into a fraction
$\log_2\left(\frac{x+3}{x+5}\right)=1$	We used property #7. Now change into exponential form.
$\frac{x+3}{x+5}=2^1$	We can solve this by cross multiplying.
$2(x+5)=x+3$	Now solve for x.
$2x+10=x+3$	
$x=-7$	

If we put -7 into the original we get $\log_2(-7+3)-\log_2(-7+5)=1$ which is $\log_2(-4)+\log_2(-2)=1$ We can't have a negative inside the log, so we reject the answer x = -7. Since our only answer did not work, the answer to the problem is "no solution" or undefined.