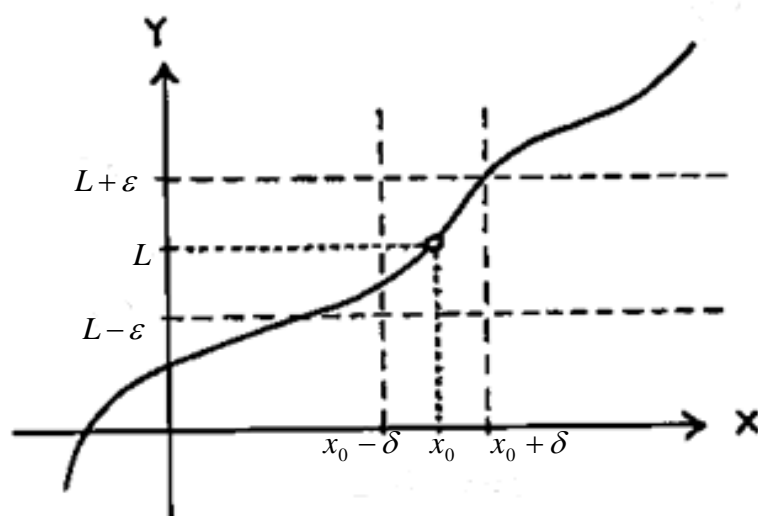


2.2 The Precise Definition of a Limit

Precise definition of a limit

Now we will look at the definition of a limit. Suppose we want to find $\lim_{x \rightarrow x_0} f(x)$. We are going to let δ represent an increment. This will tell us how close we are to x_0 . We might be, say, 0.001 off from x_0 with some measuring error. We will let L represent the actual value of the limit. Now depending how much we are off from our measurement of c this will affect the values that we get for the limit. We might be off on this measurement as well. Let's call ε the amount we are off from the limit. So, δ corresponds to x values and ε represents y values. See the picture on the next page. This shows the relationship between all these variables we just mentioned.

What this is saying below is how close to x_0 does x have to be so that $f(x)$ differs from L by less than ε units?



Now that we have defined these variables, let's look at the definition of the limit.

Let f be defined on an open interval containing c and let L be a real number. Then:

$\lim_{x \rightarrow x_0} f(x) = L$ means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$.

What this means is that if we say an error in how far we are from the actual limit ε , then there is also an error coming from the x -direction, which is δ . The positive distance between the x value and x_0 has to be larger than zero but less than δ . If this is true then the difference between the actual y -value of the limit and the y -value of the error $f(x)$ must be greater than zero, but less than ε .

EXAMPLE: Sketch the interval (a, b) on the x -axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x , $0 < |x - x_0| < \delta$ such that $a < x < b$. Given: $a = 1$, $b = 7$, $x_0 = 2$.

You would sketch this interval on a number line, you will get: $(1 \underline{2} \underline{\quad} 7)$. The value of δ is the difference between x_0 and the CLOSEST endpoint. If we take $x_0 - a$, we get 1. If we take $b - x_0$, we get 5. Therefore δ would have to be the smaller of these two results, so $\delta = 1$.

EXAMPLE: Use the ε - δ definition of a limit to prove that $\lim_{x \rightarrow 3} 5x - 4 = 11$.

Before we write the proof, let's define our variables. $x_0 = 3$, $L = 11$, and $f(x) = 5x - 4$. The following rest of what follows is the answer. This answer is actually a series of statements that make up a proof. Here we go:

Proof:

For each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $|5x - 4 - 11| < \varepsilon$.

$$|(5x - 4) - 11| < \varepsilon$$

$$|5(x - 3)| < \varepsilon$$

$$5|x - 3| < \varepsilon \text{ so } |x - 3| < \frac{\varepsilon}{5}. \text{ Hence, let } \delta = \frac{\varepsilon}{5}.$$

$$\text{Hence, if } 0 < |x - 3| < \delta = \frac{\varepsilon}{5}$$

$$|x - 3| < \frac{\varepsilon}{5}, \text{ or}$$

$$5|x - 3| < \varepsilon, \text{ or}$$

$$|5x - 15| < \varepsilon, \text{ or}$$

$$|(5x - 4) - 11| < \varepsilon, \text{ or}$$

$$|f(x) - L| < \varepsilon$$

EXAMPLE: Use the ε - δ definition of a limit to prove that $\lim_{x \rightarrow 4} \frac{x}{2} + 6 = 8$.

Before we write the proof, let's define our variables. $x_0 = 4$, $L = 8$, and $f(x) = \frac{x}{2} + 6$. The following rest of what follows is the answer. This answer is actually a series of statements that make up a proof. Here we go:

Proof:

For each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - 4| < \delta$, then $\left| \frac{x}{2} + 6 - 8 \right| < \varepsilon$.

$$\left| \frac{x}{2} + 6 - 8 \right| < \varepsilon$$

$$\text{Hence, if } 0 < |x - 4| < \delta = 2\varepsilon$$

$$\left| \frac{x}{2} - 2 \right| < \varepsilon$$

$$|x - 4| < 2\varepsilon$$

$$\left| \frac{1}{2}(x - 4) \right| < \varepsilon$$

$$\frac{1}{2}|x - 4| < \varepsilon$$

$$\frac{1}{2}|x - 4| < \varepsilon$$

$$\left| \frac{1}{2}(x - 4) \right| < \varepsilon, \text{ so } \left| \frac{x}{2} - 2 \right| < \varepsilon \text{ and } \left| \frac{x}{2} + 6 - 8 \right| < \varepsilon \text{ and}$$

$$|x - 4| < 2\varepsilon$$

$$|f(x) - L| < \varepsilon$$

Hence, let $\delta = 2\varepsilon$.

EXAMPLE: In the following exercises, find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give the largest value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds:

a.) $f(x) = 2x - 2$, $L = -6$, $x_0 = -2$, $\varepsilon = 0.02$

$$|2x - 2 - (-6)| < 0.02 \quad \text{First we plug in our values into } |f(x) - L| < \varepsilon. \text{ Then we solve for } x.$$

$$|2x + 4| < 0.02$$

$$-0.02 < 2x + 4 < 0.02$$

$$-4.02 < 2x < -3.98$$

$$-2.01 < x < -1.99$$

$$(-2.01, -1.99)$$

Here is our open interval answer.

For the second part of the question, let's first start with $0 < |x - x_0| < \delta$. We are given that $x_0 = -2$, so this becomes $0 < |x + 2| < \delta$. Now let's solve this for x . We know that absolute values are greater than 0, so this problem can be simplified to: $|x + 2| < \delta$. Now let's solve this for x : $-\delta < x + 2 < \delta$. Now subtract 2 from both sides to get: $-2 - \delta < x < \delta - 2$. From above, we solved for x and got: $-2.01 < x < -1.99$. Therefore we know that $-2 - \delta = -2.01$ or $\delta - 2 = -1.99$. Solving either one of these will get $\delta = 0.01$. Instead of doing all of this, we could have just noticed that the interval $-2.01 < x < -1.99$ is centered around our x_0 , which was 2. We notice that the distance from either endpoint to the number 2 is 0.01, so we know $\delta = 0.01$.

b.) $f(x) = \sqrt{x - 7}$, $L = 4$, $x_0 = 23$, $\varepsilon = 1$

$$|\sqrt{x - 7} - 4| < 1 \quad \text{First we plug in our values into } |f(x) - L| < \varepsilon. \text{ Then we solve for } x.$$

$$-1 < \sqrt{x - 7} - 4 < 1$$

$$3 < \sqrt{x - 7} < 5$$

Now square both sides.

$$9 < x - 7 < 25$$

$$16 < x < 32$$

$$(16, 32)$$

Here is our open interval answer.

When you do problems involving square roots, we notice that our interval is not centered around our x_0 like last time. We are given that $x_0 = 23$, so this becomes $0 < |x - 23| < \delta$. Now let's solve this for x . We know that absolute values are greater than 0, so this problem can be simplified to: $|x - 23| < \delta$. Now let's solve this for x : $-\delta < x - 23 < \delta$. Now add 23 to both sides to get: $23 - \delta < x < \delta + 23$. So we have $23 - \delta = 16$ and $\delta + 23 = 32$. Solving both of these we get $\delta = 7$ and $\delta = 9$. You want to minimize the error in the x direction, so we want the smaller answer. Therefore, our answer is going to be $\delta = 7$.

c.) $f(x) = 1/x$, $L = -1$, $x_0 = -1$, $\varepsilon = 0.1$

$$|1/x + 1| < 0.1$$

First we plug in our values into $|f(x) - L| < \varepsilon$. Then we solve for x .

$$-0.1 < 1/x + 1 < 0.1$$

Let's write this in terms of fractions.

$$-\frac{1}{10} < 1/x + 1 < \frac{1}{10}$$

Now subtract 1 from all sides.

$$-\frac{11}{10} < 1/x < -\frac{9}{10}$$

Next we need to take the reciprocal of both sides and reverse the inequality.

$$-\frac{10}{11} > x > -\frac{10}{9}$$

$$\left(-\frac{10}{9}, -\frac{10}{11}\right)$$

Here is our open interval answer.

When you do problems involving square roots, we notice that our interval is not centered around our x_0 like last time. We are given that $x_0 = -1$, so this becomes $0 < |x + 1| < \delta$. Now let's solve this for x . We know that absolute values are greater than 0, so this problem can be simplified to: $|x + 1| < \delta$. Now let's solve this for x :

$$-\delta < x + 1 < \delta. \text{ Now subtract 1 from both sides to get: } -\delta - 1 < x < \delta - 1. \text{ So we have } -\delta - 1 = -\frac{10}{9} \text{ and}$$

$$\delta - 1 = -\frac{10}{11}. \text{ Solving both of these we get } \delta = \frac{1}{9} \text{ and } \delta = \frac{1}{11}. \text{ You want to minimize the error in the } x$$

direction, so we want the smaller answer. Therefore, our answer is going to be $\delta = \frac{1}{11}$.

d.) $f(x) = x^2$, $L = 3$, $x_0 = \sqrt{3}$, $\varepsilon = 0.1$ (Round answers to four decimal places)

$$|x^2 - 3| < 0.1$$

First we plug in our values into $|f(x) - L| < \varepsilon$. Then we solve for x .

$$-0.1 < x^2 - 3 < 0.1$$

We add 3 to both sides of the equation.

$$2.9 < x^2 < 3.1$$

Now square root both sides.

$$1.7029 < x < 1.7607$$

$$(1.7029, 1.7607)$$

Here is our open interval answer.

We are given that $x_0 = \sqrt{3}$, so this becomes $0 < |x - \sqrt{3}| < \delta$. Now let's solve this for x . We know that absolute values are greater than 0, so this problem can be simplified to: $|x - \sqrt{3}| < \delta$. Now let's solve this for x :

$$-\delta < x - \sqrt{3} < \delta. \text{ Now add } \sqrt{3} \text{ to both sides to get: } \sqrt{3} - \delta < x < \delta + \sqrt{3}. \text{ As decimals this can be written as: } 1.7321 - \delta < x < \delta + 1.7321. \text{ So we have } 1.7321 - \delta = 1.7029 \text{ and } \delta + 1.7321 = 1.7607. \text{ Solving these we get } \delta = 0.0292 \text{ and } \delta = 0.0286, \text{ so we want the smaller answer. Therefore } \delta = 0.0286.$$