

3.10 Related Rates

This section involves word problems dealing with how something is changing with respect to time. We will first find a formula that is used for the problem and then we will take the derivative of it with respect to time. Before we get to the part where we come up with the equation, let's first practice taking a derivative implicitly with respect to time.

EXAMPLE: If $V = 2(x^2 - 3y)$, find $\frac{dV}{dt}$ by using implicit differentiation. Then find $\frac{dV}{dt}$ when $x = 3$ given

$$\frac{dx}{dt} = -1 \text{ and } \frac{dy}{dt} = 4.$$

First we will distribute the 2 to get $V = 2x^2 - 6y$. We want to take the derivative of both sides with respect to t . When we take the derivative of both sides with respect to t you will get:

$$\frac{dV}{dt} = 4x \frac{dx}{dt} - 6 \frac{dy}{dt}$$

The reason why we need the $\frac{dx}{dt}$ and $\frac{dy}{dt}$ is because we are taking the derivative with respect to t . This is the same as using implicit differentiation. In the previous section we needed $\frac{dy}{dx}$ because we were taking the derivative with respect to x and we did not know what y is. In this case x and y are unknown variables because everything is with respect to t .

Now we need to substitute the values that were given. You will get:

$$\frac{dV}{dt} = 4x \frac{dx}{dt} - 6 \frac{dy}{dt}$$

$$\frac{dV}{dt} = 4(3)(-1) - 6(4)$$

$$\frac{dV}{dt} = -36.$$

EXAMPLE: If $x^2 + y^3 = 15$, find $\frac{dy}{dt}$ when $x = 3$, $y = 4$, and $\frac{dx}{dt} = 8$.

First we need to start with our equation and take the derivative of both sides implicitly with respect to x :

$$2x \frac{dx}{dt} + 3y^2 \frac{dy}{dt} = 0$$

Now we substitute our given values.

$$2(3)(8) + 3(4)^2 \frac{dy}{dt} = 0$$

Now solve for $\frac{dy}{dt}$.

$$48 = -48 \frac{dy}{dt}, \text{ so } \frac{dy}{dt} = -1.$$

EXAMPLE: A point is moving along the graph of $y = \frac{1}{1+x^2}$ such that the point's movement in the x direction is 2 cm/sec. Find how fast the point is moving in the y direction when $x = -2$?

We need to take the derivative of $y = \frac{1}{1+x^2}$ implicitly with respect to t. First let's rewrite the equation as: $y = (1+x^2)^{-1}$. This way we don't need to use the quotient rule. When we take the derivative we will get:

$$\frac{dy}{dt} = -(1+x^2)^{-2} \cdot 2x \frac{dx}{dt} \quad \text{Which can be rewritten as:} \quad \frac{dy}{dt} = \frac{-2x}{(1+x^2)^2} \cdot \frac{dx}{dt}.$$

Now we need to substitute the given information. It says that the point's movement in the x direction is 2 cm/sec. This means $\frac{dx}{dt} = 2$ since this is the rate at which x is changing with respect to time. We are given that $x = -2$. The question is asking us to find how fast the point is moving in the y direction. They are asking for $\frac{dy}{dt}$. Let's put in the information:

$$\frac{dy}{dt} = \frac{-2x}{(1+x^2)^2} \cdot \frac{dx}{dt} \quad \text{and after putting in the given information we get} \quad \frac{dy}{dt} = \frac{-2(-2)}{(1+(-2)^2)^2} \cdot (2).$$

We can simplify this to $\frac{dy}{dt} = \frac{4}{(5)^2} \cdot (2)$ and finally we get $\frac{dy}{dt} = \frac{8}{25}$.

EXAMPLE: A stone is dropped into a puddle causing ripples in the form of concentric circles. The radius, r, of the outer ripple is increasing at a rate of 2 cm/sec. When the radius is 0.5 meters, at what rate is the total area, A, of the disturbed water changing with respect to time? Note, the area of a circle is $A = \pi \cdot r^2$.

They tell us which formula to use, which is $A = \pi \cdot r^2$. We want to take the derivative of both sides with respect to t. You will get: $\frac{dA}{dt} = 2 \cdot \pi \cdot r \frac{dr}{dt}$. They tell us the radius is increasing at a rate of 2 cm/sec. This means that $\frac{dr}{dt} = 2$. We also know the radius. Since $\frac{dr}{dt}$ is in terms of centimeters, we need to convert 0.5 meters to centimeters. So 0.5 meters = 50 centimeters since we move the decimal 2 places to the right. When we substitute this into our formula we get $\frac{dA}{dt} = 2 \cdot \pi \cdot 50 \cdot 2$, or $\frac{dA}{dt} = 200\pi \text{ cm}^2 / \text{sec}$.

EXAMPLE: The volume of a sphere is $V = \frac{4}{3}\pi \cdot r^3$. If the radius is changing at a rate of 2 in/min, find the rate the volume is changing when the radius is 6 inches.

One again they tell us what formula to use, which is $V = \frac{4}{3}\pi \cdot r^3$. We need to take the derivative of both sides with respect to t. You will get: $\frac{dV}{dt} = 4\pi \cdot r^2 \frac{dr}{dt}$. Now we need to put in the given information. They tell us the radius is changing at a rate of 2 in/min, which means $\frac{dr}{dt} = 2$. We also know $r = 6$. Putting this in we get: $\frac{dV}{dt} = 4\pi \cdot 6^2 (2)$, so $\frac{dV}{dt} = 288\pi$.

EXAMPLE: A conveyor belt drops sand into a conical pile whose radius is always twice the height. Sand falls at a rate of 5 cubic meters per second. How fast is its height increasing when the radius is 20 meters? (Note: you will need to use the volume of cone formula, which is $V = \frac{1}{3}\pi \cdot r^2 h$).

This problem has too many variables. They eventually ask us for how much the height is increasing, so we need to make sure our formula only is in terms of h . They do tell us that the radius is twice the height, so $r = 2h$.

We can put this into our volume formula to get $V = \frac{1}{3}\pi \cdot (2h)^2 h$, or $V = \frac{4}{3}\pi \cdot h^3$. When we take the derivative

of both sides you will get $\frac{dV}{dt} = 4\pi \cdot h^2 \frac{dh}{dt}$. If the radius is 20 meters then the height must be 10 meters since

the radius is always twice the height. We also know that sand is falling at a rate of 5 cubic meters per second.

This falling sand is adding to our volume, so $\frac{dV}{dt} = 5$. We want to put these values into our formula and solve

for $\frac{dh}{dt}$. We will have $5 = 4\pi \cdot 10^2 \frac{dh}{dt}$. By solving for $\frac{dh}{dt}$ we will get $\frac{dh}{dt} = \frac{5}{4\pi \cdot 10^2}$, so $\frac{dh}{dt} = \frac{1}{80\pi}$ m/s.

EXAMPLE: The length L of a rectangle is decreasing at the rate of 2 cm/sec while the width W is increasing at the rate of 2 cm/sec. When $L = 12$ cm and $W = 5$ cm, find the rates of change of a) the area, b.) the perimeter, and c.) the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

a.) The area of a rectangle is $A = LW$. We will take the derivative of this with respect to time. We will get:

$\frac{dA}{dt} = L \frac{dW}{dt} + W \frac{dL}{dt}$. Notice a product rule was needed here. The problem tells us that $\frac{dL}{dt} = -2$ and $\frac{dW}{dt} = 2$,

$L = 12$ and $W = 5$. We can substitute this: $\frac{dA}{dt} = (12)(2) + (5)(-2)$. Our answer is $\frac{dA}{dt} = 14$ cm^2 / sec . This means the area is increasing at this moment.

b.) The perimeter of a rectangle is $P = 2L + 2W$. Once again we will take the derivative with respect to time.

$\frac{dP}{dt} = 2 \frac{dL}{dt} + 2 \frac{dW}{dt}$. A product rule was NOT needed here. The problem tells us that $\frac{dL}{dt} = -2$ and $\frac{dW}{dt} = 2$,

so we can substitute this into our expression: $\frac{dP}{dt} = 2(-2) + 2(2)$. We get $\frac{dP}{dt} = 0$ cm/sec. This means the perimeter is constant (not changing) at this moment.

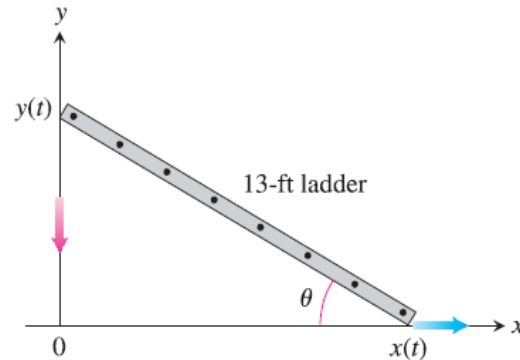
c.) For the diagonals, we need to use the Pythagorean Theorem: $D^2 = L^2 + W^2$. Now we take the derivative of both sides with respect to time: $2D \frac{dD}{dt} = 2L \frac{dL}{dt} + 2W \frac{dW}{dt}$. If we use $D^2 = L^2 + W^2$ with $L = 12$ and $W = 5$

we get $D^2 = 12^2 + 5^2$, so $D = 13$. We also know $\frac{dL}{dt} = -2$ and $\frac{dW}{dt} = 2$, $L = 12$ and $W = 5$. Now plug all this

into our formula: $2(13) \frac{dD}{dt} = 2(12)(-2) + 2(5)(2)$. This gives us $26 \frac{dD}{dt} = -28$. When we solve this, we get

EXAMPLE: A 13-foot ladder is leaning against a house when its base starts to slide away. By the time the base is 12 feet from the house, the base is moving at a rate of 5 ft/sec. a.) Find the rate at which the ladder is falling down the side of the house. b.) Consider the triangle formed by the ladder, house, and ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 12 feet from the base of the house.

First let's draw a picture to describe the situation:



Let's first look at part a.

Notice that there is a right triangle formed here. This means we can use $a^2 + b^2 = c^2$ as our equation. Our equation will be: $x^2 + y^2 = 13^2$, or $x^2 + y^2 = 169$. Now we will take the derivative of both sides of this equation with respect to time. You will have $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$. Let's first do part a. It says that the base of the ladder is moving at a rate of 5 ft/sec. This means $\frac{dx}{dt} = 5$. It also tells us that the base is 12 feet from the house. So that means we need to find the y. In our formula $x^2 + y^2 = 169$ we can put in a 12 for x and solve for y. You will get: $12^2 + y^2 = 169$, so by solving this we get $y = 5$. Now we can put all this information into our formula, $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$. You will get $2(12)(5) + 2(5) \frac{dy}{dt} = 0$. Solving for $\frac{dy}{dt}$ we get $120 = -10 \frac{dy}{dt}$ so $\frac{dy}{dt} = -12$ ft/sec. The negative means the ladder is moving down.

Now let's look at part b. The area of a triangle is $A = \frac{1}{2}bh$, or in our case we have $A = \frac{1}{2}xy$. When we take the derivative of the right hand side we need to use the product rule. You will get:

$$\frac{dA}{dt} = \frac{1}{2}x \frac{dy}{dt} + y \cdot \frac{1}{2} \cdot \frac{dx}{dt}.$$

We know that $x = 12$ and $y = 5$ from the previous part. Also $\frac{dx}{dt} = 5$ from above and $\frac{dy}{dt} = -12$, which was our result from part a. Putting this all in we get:

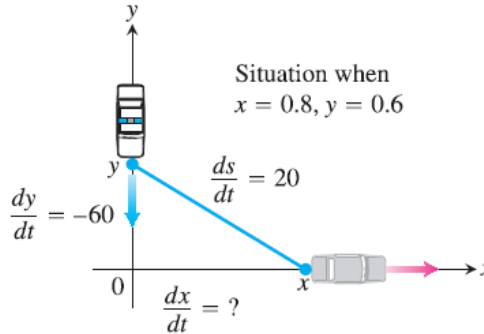
$$\frac{dA}{dt} = \frac{1}{2}(12)(-12) + (5) \cdot \frac{1}{2} \cdot (5),$$

so after simplifying we get: $\frac{dA}{dt} = \frac{-119}{2}$ ft²/sec. The negative here means that the area is decreasing, which makes sense since the ladder is falling and creating a smaller triangle.

More on next page.

EXAMPLE: A police cruiser, approaching a right-angled intersection from the north is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 miles north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60mph at the instant of measurement, what is the speed of the car?

Let's look at the picture for this one:



Let s be the distance between the two cars. We have another right triangle here so we need to use $a^2 + b^2 = c^2$. Our formula will be $x^2 + y^2 = s^2$. We need to take the derivative of both sides with respect to t . This is:

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2s \frac{ds}{dt}$$

If $x = 0.8$ and $y = 0.6$, then we can find s by plugging these values into $x^2 + y^2 = s^2$.

We have $0.8^2 + 0.6^2 = s^2$. Solving this we will get $s = 1$. We also know that $\frac{dy}{dt} = -60$ since the car is moving south, or down as our picture states. In addition, $\frac{ds}{dt} = 20$ since this is how much the distance between the cars

is changing. We will put all of these known values in to the formula $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2s \frac{ds}{dt}$. It will look like:

$$2(0.8) \frac{dx}{dt} + 2(0.6)(-60) = 2(1)(20)$$

Solving for $\frac{dx}{dt}$ we get $1.6 \frac{dx}{dt} - 72 = 40$, and so $\frac{dx}{dt} = 70$ mph.

EXAMPLE: A hot air balloon rising straight up from a level field is tracked by a range finder 500 feet from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$ the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Once again we need to look at a picture of what's going on:

Since we are talking about angles we probably need to use trigonometry on this one. We want to find a trig function that relates the opposite and adjacent sides. This will be tangent. We will use the tangent definition to get our formula which will be: $\tan \theta = \frac{y}{500}$. We can solve for y to get $y = 500 \tan \theta$.

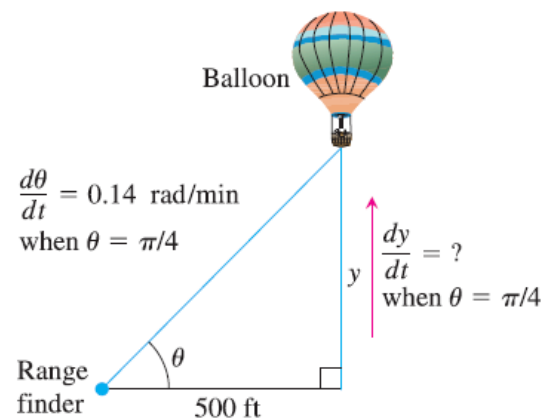
Now take the derivative of both sides with respect to t .

$$\frac{dy}{dt} = 500 \sec^2 \theta \cdot \frac{d\theta}{dt}$$

We know $\theta = \frac{\pi}{4}$ and $\frac{d\theta}{dt} = 0.14$. Putting these into the equation we will get

$$\frac{dy}{dt} = 500 \sec^2 \left(\frac{\pi}{4} \right) \cdot 0.14$$

We get $\frac{dy}{dt} = 500(\sqrt{2})^2 \cdot 0.14$ so then $\frac{dy}{dt} = 140$ ft/min.



EXAMPLE: A man 6 ft tall walks at the rate of 3 ft/s toward a streetlight that is 15 ft above the ground. **(a)** At what rate is the tip of his shadow moving? **(b)** At what rate is the length of his shadow decreasing when he is 40 ft from the base of the light?

(a) For this one, we should draw a diagram. The diagram below has a smaller right triangle inside a bigger right triangle. We will call x the distance from the base of the streetlight to the tip of the shadow and y the distance from the man to the base of the street light:

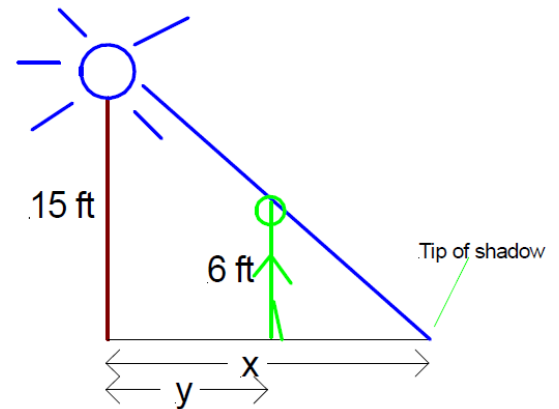
Based on our diagram, the rate at which is the tip of the shadow is moving is $\frac{dx}{dt}$. We are given $\frac{dy}{dt} = 3$ ft/sec. We can get a relationship between x and y by using similar triangles. Note that the base of the smaller triangle is $x - y$. The relationship is $\frac{15}{x} = \frac{6}{x - y}$. If you cross multiply you will get $15x - 15y = 6x$.

We want to solve for x , so $9x = 15y$ and $x = \frac{5}{3}y$. Now we want to take the derivative of both sides with respect to t . You will get:

$$\frac{dx}{dt} = \frac{5}{3} \frac{dy}{dt}. \text{ Since we are given } \frac{dy}{dt} = 3 \text{ ft/sec, then } \frac{dx}{dt} = \frac{5}{3}(3) = 5 \text{ ft/sec.}$$

(b) Let s represent the length of the man's shadow. Then we have this relationship: $s = x - y$. We will take the derivative of both sides with respect to t : $\frac{ds}{dt} = \frac{dx}{dt} - \frac{dy}{dt}$. We know that $\frac{dx}{dt} = 5$ ft/sec and $\frac{dy}{dt} = 3$ ft/sec. So

plug these in the get the answer: $\frac{ds}{dt} = 5 - 3$. So the final answer is $\frac{ds}{dt} = 2$ ft/sec.



Notice that in both of these cases as long as the man is moving at a constant rate, the rate of his shadow will also remain constant regardless of how far away he is from the street light.