

## 3.3 Differentiation Rules

The derivative has several different types of notation:

Notation	Meaning
$f'(x)$	Derivative of $f(x)$
$\frac{dy}{dx}$	Derivative of $y$ with respect to $x$ .
$y'$	Derivative of $y$
$\frac{d}{dx}[f(x)]$	Derivative of $f$ with respect to $x$ .

### Constant Rule

$\frac{d}{dx}[c] = 0$  This means the derivative of any number is zero. For example, suppose we had  $y = 9$  and the question asked us to find  $y'$ . Then our answer would automatically be zero because  $y = 9$  is a horizontal line. The slope of a horizontal line is always 0.

EXAMPLE: Find  $f'(x)$  if  $f(x) = 4 \cdot \pi \cdot e^2$ .

Since  $e = 2.71\dots$  and  $\pi = 3.14\dots$  then this whole equation is a constant, so  $f'(x) = 0$ .

Can you see a pattern of the derivative with the following table?

Original function	Derivative
$y = x^2$	$y' = 2x$
$y = x^3$	$y' = 3x^2$
$y = x^5$	$y' = 5x^4$

It seems like the power comes down and the exponent is reduced by one. We can use this information to derive a formula for the derivative for  $y = x^n$ . This will be called the power rule.

### Power Rule

If  $n$  is any real number, then  $\frac{d}{dx}x^n = nx^{n-1}$  for all  $x$  where powers of  $x^n$  and  $x^{n-1}$  are defined.

### Other Derivative Rules

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$$

EXAMPLE: If  $y = x^{12}$ , find  $y'$ .

We will use the power rule formula. Here  $n = 12$ .  $y' = 12x^{12-1}$  so  $y' = 12x^{11}$ .

EXAMPLE: If  $y = 3x^4$ , find  $y'$ .

We can write  $y$  as  $y = 3 \cdot x^4$ . Now we use the power rule:  $y' = 3 \cdot 4x^3$ , so  $y' = 12x^3$

EXAMPLE: If  $g(x) = \frac{3}{2}x^6 - x + 3$ , find  $g'(x)$ .

First we will rewrite this as  $g(x) = \frac{3}{2} \cdot x^6 - 1x^1 + 3x^0$ . For this one we can apply the power rule to each term separately:  $g'(x) = \frac{3}{2} \cdot 6x^5 - 1x^0 + 0$ . After simplifying we get:  $g'(x) = 9x^5 - 1$ .

EXAMPLE: If  $y = 3x(6x - 5x^2)$ , find  $y'$ .

I recommend distributing first. You will get:  $y = 18x^2 - 15x^3$ . Now apply the power rule. You will get:  $y' = 36x - 45x^2$ . You can also write it as  $y' = 9x(4 - 5x)$

EXAMPLE: If  $f(x) = \sqrt[4]{x}$ , find  $f'(x)$ .

This one can be rewritten with fractional exponents:  $f(x) = x^{\frac{1}{4}}$ . We still can apply the power rule here:  $f(x) = \frac{1}{4}x^{\frac{1}{4}-1}$ , so after simplifying you get  $f(x) = \frac{1}{4}x^{-\frac{3}{4}}$ . You do not want this as a negative exponent, so place  $x$  in the denominator. You will get  $f(x) = \frac{1}{4x^{\frac{3}{4}}}$ .

EXAMPLE: If  $f(x) = \frac{2}{x^3}$ , find  $f'(x)$ .

If there is a variable in the denominator, write this as a negative exponent. You will get  $f(x) = 2x^{-3}$ . Then apply the power rule. You will get  $f'(x) = -6x^{-4}$ . Now write it without negative exponents and you will have  $f'(x) = -\frac{6}{x^4}$ .

EXAMPLE: If  $f(x) = x + \frac{1}{x}$ , find  $f'(x)$ .

We need to first rewrite this as a negative exponent. You will get  $f(x) = x + x^{-1}$ . Apply the power rule. You will get  $f'(x) = 1 - x^{-2}$ . Now rewrite without negative exponents:  $f'(x) = 1 - \frac{1}{x^2}$ .

EXAMPLE: If  $f(x) = \frac{2x^2 - 3x + 1}{x}$ , find  $f'(x)$ .

I would suggest dividing each term by  $x$  and then simplifying:  $f(x) = \frac{2x^2}{x} - \frac{3x}{x} + \frac{1}{x}$  so  $f(x) = 2x - 3 + x^{-1}$ .

Now apply the power rule:  $f'(x) = 2 - x^{-2}$ . Finally rewrite with positive exponents:  $f'(x) = 2 - \frac{1}{x^2}$ .

EXAMPLE: If  $y = \sqrt{x} - \frac{5}{\sqrt{x}} + \sqrt{2}$ , find  $y'$ .

First write change the radicals into fractional exponents:  $y = x^{\frac{1}{2}} - 5x^{-\frac{1}{2}} + \sqrt{2}$ . I did not change the last square root of 2 into a fractional exponent because when I take the derivative this term will turn to zero. Now apply

the product rule:  $y' = \frac{1}{2}x^{-\frac{1}{2}} + \frac{5}{2}x^{-\frac{3}{2}}$ . Lastly rewrite with positive exponents:  $y' = \frac{1}{2x^{\frac{1}{2}}} + \frac{5}{2x^{\frac{3}{2}}}$ . You may

leave it with fraction exponents unless the question specifically tells you not to. Suppose this question did want

you to write it with radicals. Then you would have:  $y' = \frac{1}{2\sqrt{x}} + \frac{5}{2\sqrt{x^3}}$ , which is also  $y' = \frac{1}{2\sqrt{x}} + \frac{5}{2x\sqrt{x}}$ .

With common denominators it can be written as  $y' = \frac{x+5}{2x\sqrt{x}}$ .

EXAMPLE: Determine the point(s) at which  $y = x^2 + 1$  has a horizontal tangent line.

A horizontal tangent line means that the slope is zero. So we must take the derivative and set it equal to zero.

$y' = 2x$	We have our derivative. Now we must set it equal to zero.
$0 = 2x$	Now we solve for $x$ .
$x = 0$	So when $x = 0$ , $y = 1$ , so the point is $(0, 1)$ .

EXAMPLE: Determine the point(s) at which  $y = x^3 - 27x$  has a horizontal tangent line.

$y' = 3x^2 - 27$	We have our derivative. Now we must set it equal to zero.
$0 = 3x^2 - 27$	Now we solve for $x$ .
$27 = 3x^2$	
$9 = x^2$	
$x = \pm 3$	Put each of these into the original equation for $x$ and we get $(3, -54)$ and $(-3, 54)$ .

We unfortunately can't use the product rule on everything. We will look at ways to find derivatives of things multiplied together and divided.

**Product Rule**

$$\frac{d}{dx}[f \cdot g] = f \cdot g' + g \cdot f'$$

**Quotient Rule**

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{g \cdot f' - f \cdot g'}{g^2}$$

EXAMPLE: Use the product rule to find  $H'(x)$  if  $H(x) = (6x + 5)(x^3 - 2)$ .

In our problem, I labeled f and g. Now we apply the product rule formula. I have labeled each part.

$$H'(x) = \overset{f}{(6x + 5)} \overset{g'}{(3x^2)} + \overset{g}{(x^3 - 2)} \overset{f'}{(6)} \quad \text{Now we just simplify.}$$

$$H'(x) = 18x^3 + 15x^2 + 6x^3 - 12$$

$$H'(x) = 24x^3 + 15x^2 - 12$$

EXAMPLE: Use the quotient rule to find  $H'(x)$  if  $H(x) = \frac{x^2 + 2}{2x - 7}$ .

Here the numerator is always f and the denominator is always g. Now we use the quotient rule:

$$H'(x) = \frac{\overset{g}{(2x - 7)} \overset{f'}{(2x)} - \overset{f}{(x^2 + 2)} \overset{g'}{(2)}}{\overset{g^2}{(2x - 7)^2}} \quad \text{Now we distribute and simplify.}$$

$$H'(x) = \frac{4x^2 - 14x - 2x^2 - 4}{(2x - 7)^2} \quad \text{Now just simplify to get our answer: } H'(x) = \frac{2x^2 - 14x - 4}{(2x - 7)^2}$$

EXAMPLE: Use the product rule to find  $M'(x)$  if  $M(x) = \sqrt{x}(4 - x^2)$ . Write answer as a single fraction.

Once again f and g are labeled. We need to use the product rule again.

$$M'(x) = \overset{f}{\sqrt{x}} \overset{g'}{(-2x)} + \overset{g}{(4 - x^2)} \overset{f'}{\frac{1}{2}x^{-\frac{1}{2}}} \quad \text{Now get rid of the negative exponent and rewrite with radicals.}$$

$$M'(x) = -2x\sqrt{x} + \frac{4-x^2}{2\sqrt{x}}$$

We need common denominators to write this as a single fraction.

$$M'(x) = \frac{-2x\sqrt{x}}{1} \cdot \left(\frac{2\sqrt{x}}{2\sqrt{x}}\right) + \frac{4-x^2}{2\sqrt{x}}$$

We need to multiply across the top and across the bottom.

$$M'(x) = \frac{-4x^2}{2\sqrt{x}} + \frac{4-x^2}{2\sqrt{x}}$$

Now combine as a single fraction.

$$M'(x) = \frac{4-5x^2}{2\sqrt{x}}$$

EXAMPLE: Use the product rule to find  $M'(x)$  if  $M(x) = (x^2 + 1)(x + 5 + 1/x)$ .

$f$        $g$

$f$        $g'$        $g$        $f'$

$$M'(x) = (x^2 + 1)\left(1 - \frac{1}{x^2}\right) + \left(x + 5 + \frac{1}{x}\right)(2x)$$

Now multiply this together.

$$M'(x) = x^2 - 1 + 1 - \frac{1}{x^2} + 2x^2 + 10x + 2$$

We simplify and add like terms.

$$M'(x) = 3x^2 + 10x + 2 - \frac{1}{x^2}$$

We can leave our answer like this.

EXAMPLE: Use the quotient rule to find  $H'(x)$  if  $H(x) = \frac{x}{\sqrt{x}-1}$ .

We will use the quotient rule again. Remember the numerator is always  $f$  and the denominator is always  $g$ .

$$H'(x) = \frac{(\sqrt{x}-1)(1) - x \cdot \frac{1}{2}x^{-\frac{1}{2}}}{(\sqrt{x}-1)^2}$$

Now we will simplify. The  $x$ 's cancel this way:  $x \cdot x^{-\frac{1}{2}} = x^{\frac{1}{2}}$ .

$$H'(x) = \frac{\sqrt{x}-1-\frac{1}{2}x^{\frac{1}{2}}}{(\sqrt{x}-1)^2}$$

Now we will multiply the top and bottom by 2 to clear the fraction.

$$H'(x) = \frac{2}{2} \cdot \frac{\sqrt{x}-1-\frac{1}{2}x^{\frac{1}{2}}}{(\sqrt{x}-1)^2}$$

$$H'(x) = \frac{2\sqrt{x}-2-\sqrt{x}}{2(\sqrt{x}-1)^2}$$

I multiplied by 2 and then changed the fractional power back into a radical.

$$H'(x) = \frac{\sqrt{x}-2}{2(\sqrt{x}-1)^2}$$

I added the like terms on top and this is as far as we can go.

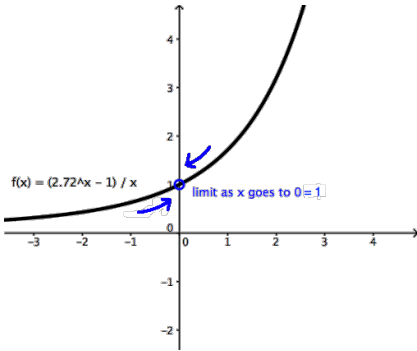
**Derivative of  $e^x$** 

We want to derive the derivative of  $e^x$ . So far the only way we can do this is using the limit process. First we will start with the definition:  $\frac{d}{dx}[e^x] = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ . In this case,  $f(x) = e^x$ , and  $f(x+h) = e^{x+h}$ . Let's

substitute this into the limit definition:  $\frac{d}{dx}[e^x] = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}$ . We can rewrite this as:  $\frac{d}{dx}[e^x] = \lim_{h \rightarrow 0} \frac{e^x e^h - e^x}{h}$ .

Factor out  $e^x$ :  $\frac{d}{dx}[e^x] = \lim_{h \rightarrow 0} \frac{e^x(e^h - 1)}{h}$ . Then we can rewrite as:  $\frac{d}{dx}[e^x] = e^x \cdot \lim_{h \rightarrow 0} \frac{e^h - 1}{h}$ . So now let's look at

the graph of  $y = \frac{e^x - 1}{x}$  (on next page).



So we see here from the graph that  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$ . Therefore  $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$ .

So in  $\frac{d}{dx}[e^x] = e^x \cdot \lim_{h \rightarrow 0} \frac{e^h - 1}{h}$ , we can substitute 1 for  $\lim_{h \rightarrow 0} \frac{e^h - 1}{h}$ . Then we

get:  $\frac{d}{dx}[e^x] = e^x(1)$ .

Therefore,  $\frac{d}{dx}[e^x] = e^x$ . So the derivative of  $e^x$  is itself!

EXAMPLE: Given:  $y = e^{-x}$  find  $y'$ .

We have just seen the derivative of  $e^x$ , however  $e^{-x}$  often comes up in problems, so let's first find this derivative. First let's rewrite our equation as  $y = \frac{1}{e^x}$ . So now we need to use the quotient rule:

$y' = \frac{e^x(0) - 1(e^x)}{(e^x)^2}$ . Now let's simplify:  $y' = \frac{-e^x}{e^{2x}} = -\frac{1}{e^x} = -e^{-x}$ . So now we can say  $\frac{d}{dx}[e^{-x}] = -e^{-x}$ .

EXAMPLE: Given:  $y = \frac{e^x - e^{-x}}{2}$  find  $y'$ .

I will first divide each term by 2:  $y = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$ . Now we can take the derivative of each term separately:

$y' = \frac{1}{2}e^x - \frac{1}{2}(-e^{-x})$ . Simplifying we get:  $y' = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$ . Notice I used  $\frac{d}{dx}[e^{-x}] = -e^{-x}$  from above.

EXAMPLE: Given:  $y = x^2 e^x$  find  $y'$ .

For this one we need to use the product rule. The derivative of  $e^x$  is  $e^x$  as we proved earlier:

$y' = x^2 e^x + e^x(2x)$ . This can be simplified:  $y' = x^2 e^x + 2x e^x$ . You could also factor this one:  $y' = x e^x(x + 2)$ .

EXAMPLE: Given:  $y = (9x^2 - 6x + 2)e^x$  find  $y'$ .

For this one we also need to use the product rule:  $y' = (9x^2 - 6x + 2)e^x + e^x(18x - 6)$ . We can factor out the  $e^x$  from both terms:  $y' = e^x(9x^2 - 6x + 2 + 18x - 6)$ . After simplifying you will get:  $y' = e^x(9x^2 + 12x - 4)$ .

EXAMPLE: Given:  $y = \sqrt[11]{x^3} - e^3 + x^e$  find  $y'$ .

First let's rewrite this one:  $y = x^{\frac{3}{11}} - e^3 + x^e$ . To find the derivative of the first term we will use the power rule. For the second term, the derivative is zero because  $e^3$  is a constant. For the last term we can still use the power rule:  $y' = \frac{3}{11}x^{-\frac{8}{11}} - 0 + ex^{e-1}$ . This simplifies to:  $y' = \frac{3}{11x^{\frac{8}{11}}} + ex^{e-1}$ . Nothing more we can do here.

EXAMPLE: Given:  $y = \frac{4e^x}{2x^5 - 3e^x}$  find  $y'$ .

This problem requires the quotient rule:  $y = \frac{(2x^5 - 3e^x)4e^x - 4e^x(10x^4 - 3e^x)}{(2x^5 - 3e^x)^2}$ . We can factor the out the  $4e^x$

from the top:  $y' = \frac{4e^x(2x^5 - 3e^x - (10x^4 - 3e^x))}{(2x^5 - 3e^x)^2}$ . Distribute the negative:  $y' = \frac{4e^x(2x^5 - 3e^x - 10x^4 + 3e^x)}{(2x^5 - 3e^x)^2}$ .

Now simplify:  $y' = \frac{4e^x(2x^5 - 10x^4)}{(2x^5 - 3e^x)^2}$ . Finally we can do one more factoring step:  $y' = \frac{8x^4e^x(x - 5)}{(2x^5 - 3e^x)^2}$ .

## Higher Order Derivatives

$f(x)$	This is our original function
$f'(x)$	First derivative of $f$
$f''(x)$	Second derivative of $f$ (derivative of $f'(x)$ )
$f'''(x)$	Third derivative of $f$ (derivative of $f''(x)$ )
$f^{(n)}(x)$	The $n$ th derivative of $f$ (derivative of $f^{(n-1)}(x)$ )

A function has derivatives of all orders once the derivative reaches zero.

EXAMPLE: Let  $f(x) = 4x^3 + 5x^2 + 3x + 1$ . Find the derivatives of all orders.

We will use the power rule for this.

$f'(x) = 12x^2 + 10x + 3$  In order to find  $f''(x)$ , take the derivative of  $f'(x)$  using the power rule.

$f''(x) = 24x + 10$  Now we will take the derivative of  $f''(x)$  to get  $f'''(x)$ .

$f'''(x) = 24$

$f^{(4)}(x) = 0$  The derivative is zero, and all subsequent derivatives are zero, so we've found the derivatives of all orders.

EXAMPLE: Let  $f(x) = \frac{x^3 + 2x^2 - 1}{x}$ . Find  $f'''(x)$ .

You might think that we need to use the quotient rule for this one, but we don't need to. We can simplify this

by dividing each term by  $x$ :  $f(x) = \frac{x^3}{x} + \frac{2x^2}{x} - \frac{1}{x}$ . This simplifies to  $f(x) = x^2 + 2x - x^{-1}$

$$f(x) = x^2 + 2x - x^{-1}$$

$$f'(x) = 2x + 2 + x^{-2}$$

$$f''(x) = 2 - 2x^{-3}$$

$$f'''(x) = 6x^{-4}$$

$$f'''(x) = \frac{6}{x^4}$$

Take the first derivative using the power rule.

In order to find  $f''(x)$ , take the derivative of  $f'(x)$  using the power rule.

Now we will take the derivative of  $f''(x)$  to get  $f'''(x)$ .

EXAMPLE: Given:  $y = \left(\frac{x^3 - 2}{5x}\right)\left(\frac{x^2 + 5}{x^3}\right)$  find  $y'$  and  $y''$ .

Instead of applying the quotient rule twice with the product rule, let's first multiply this out:

$y = \frac{x^5 + 5x^3 - 2x^2 - 10}{5x^4}$ . Now we can break up this fraction:  $y = \frac{1}{5}x + x^{-1} - \frac{2}{5}x^{-2} - 2x^{-4}$ . To find the first

derivative we will use the power rule:  $y' = \frac{1}{5} - x^{-2} + \frac{4}{5}x^{-3} + 8x^{-5}$ . We will take the derivative of this to find the

second derivative:  $y'' = 2x^{-3} - \frac{12}{5}x^{-4} - 40x^{-6}$ . So for our final answers we will write:  $y' = \frac{1}{5} - \frac{1}{x^2} + \frac{4}{5x^3} + \frac{8}{x^5}$

and  $y'' = \frac{2}{x^3} - \frac{12}{5x^4} - \frac{40}{x^6}$ .

EXAMPLE: Given:  $y = 4x^3e^{-x}$  find  $y'$  and  $y''$ .

First we will find the first derivative using the product rule:  $y' = 4x^3(-e^{-x}) + e^{-x}(12x^2)$ . This simplifies to:

$y' = -4x^3e^{-x} + 12x^2e^{-x}$ . We want to find the derivative of this answer to get the second derivative. Each term

requires use of the product rule:  $y'' = -4x^3(-e^{-x}) + e^{-x}(-12x^2) + 12x^2(-e^{-x}) + e^{-x}(24x)$ . Now simplify:

$y'' = 4x^3e^{-x} - 24x^2e^{-x} + 24xe^{-x}$ . Then we can factor:  $y'' = 4xe^{-x}(x^2 - 6x + 6)$ . Also  $y' = 4x^2e^{-x}(3 - x)$ .