3.4 The Derivative as Rates of Change

In an earlier section we introduced average and instantaneous rates of change. In this section we will further discuss other applications in which derivatives would model the rates at which change. In particular we will discuss position, velocity, and acceleration. The table below will illustrate the relationship between these.

Notation	Meaning
$s(t)$	Position at time t
$s'(t) = v(t)$	The derivative of position is velocity
$v'(t) = a(t)$	The derivative of velocity is acceleration

EXAMPLE: Let $s(t) = 4t^2 - t + 9$ be a position function. Find the velocity and acceleration functions.

$s(t) = 4t^2 - t + 9$
$s'(t) = v(t) = 8t - 1$ We have found the velocity function by taking the first derivative of s.
$v'(t) = a(t) = 8$ We have found the acceleration function by taking the derivative of v.

EXAMPLE: An object is thrown vertically upward whose height is given by the function: $s(t) = -16t^2 + 100$ where t is in seconds and s is in feet. Find the velocity of the object after 3 seconds.

We need to find the velocity function first, which we can find by taking the derivative of s.

$s'(t) = v(t) = -32t$. Here the derivative of the position is the same as velocity, so $v(t) = -32t$. The meaning of the -32 means that every second the object is gaining a speed of 32 ft/sec as it moves down. To find out the velocity after 3 seconds, put in a 3 for t: $v(3) = -32(3)$. This will give you $v(3) = -96$ ft/sec.

EXAMPLE: Given $s(t) = \frac{t^4}{4} - t^3 + t^2$ on the interval [0, 3] where s is in meters and t is in seconds.
a.) Find the body's displacement and average velocity for the given time interval.

We want to find the body's displacement at the endpoints of the time interval and then subtract them. This will give us the displacement over the interval:

$s(0) = \frac{0^4}{4} - 0^3 + 0^2 = 0$ $s(3) = \frac{3^4}{4} - 3^3 + 3^2 = \frac{9}{4}$

Now subtract: $\frac{9}{4} - 0 = \frac{9}{4}$. So the body's displacement is $\frac{9}{4}$ meters.

In order to find the average velocity over the time interval, divide the body's displacement by the total time:

Average velocity = $\frac{9}{4}\Big/3 = \frac{3}{4}$. So the average velocity is $\frac{3}{4}$ meters per second.

b.) Find the body's speed and acceleration at the endpoints of the interval.

To find the velocity, take the derivative of the position. To find the acceleration, take the derivative of velocity.

$$v(t) = s'(t) = t^3 - 3t^2 + 2t$$
$$a(t) = v'(t) = 3t^2 - 6t + 2$$

Next we plug in the endpoints into each equation to get the answers:

$$v(0) = 0^3 - 3(0)^2 + 2(0) = 0 \ \ m/\sec \qquad v(3) = 3^3 - 3(3)^2 + 2(3) = 6 \ \ m/\sec$$
$$a(0) = 3(0)^2 - 6(0) + 2 = 2 \ \ m/\sec^2 \qquad a(3) = 3(3)^2 - 6(3) + 2 = 11 \ \ m/\sec^2$$

c.) When, if ever, during the interval does the body change direction?

In order for a body to change direction the body must first come to rest. This means the velocity would be zero. We need to find the time in which this occurs. So we will set the velocity equal to zero:

$$v(t) = t^3 - 3t^2 + 2t$$
$$0 = t^3 - 3t^2 + 2t$$
$$0 = t\left(t^2 - 3t + 2\right)$$
$$0 = t(t-1)(t-2)$$

So $t = 0$, 1, and 2. Next we will put these into a table to see which direction the body is moving. Since our interval is only between 0 and 3 this is reflected on the table below:

0 1 2 3

We need to pick test numbers to plug into the derivative. We will indicate only the sign of our answer:

+	–	+

0 1 2 3

Where we see a change in sign means there is a change in direction. So the body changes directions at $t = 1$ and and $t = 2$ seconds.

EXAMPLE: At time $t \geq 0$, the velocity of a body moving along the s-axis is $v = t^2 - 4t + 3$.

a.) Find the body's acceleration each time the velocity is zero.

First we need to find the times that the acceleration is zero: $0 = t^2 - 4t + 3$. So $0 = (t-1)(t-3)$. So $t = 1, 3$.

Next we want to find the formula for acceleration by taking the derivative of velocity: $a(t) = 2t - 4$. Now we will put in our two times:

$$a(1) = 2(1) - 4 = -2, \qquad a(3) = 2(3) - 4 = 2$$

b.) When is the body moving forward? Backward?

The body is moving forward when the velocity is positive. It is moving backward when the velocity is negative. We will set up a table using the values for t we found in part a. Then we put test values into the velocity formula. The result is below:

+	–	+

0 1 3

So we know it is moving forward on $[0,1) \cup (3,\infty)$. It is moving backward on (1,3).

c.) When is the body's velocity increasing? Decreasing?

When the velocity is increasing then there is positive acceleration. When the velocity is decreasing there is negative acceleration. We want to first set the acceleration $a(t) = 2t - 4$ equal to zero: $0 = 2t - 4$. Solving will give you $t = 2$. Now we will put 2 on a table and pick test numbers to put into the acceleration formula:

–	+

0 2

So the body's velocity is increasing on $(2,\infty)$ and decreasing on (0,2).

EXAMPLE: A dynamite blast blows a heavy rock straight up with a launch velocity of 160 feet per second (109 mph). It reaches a height of $s(t) = 160t - 16t^2$ after t seconds.

a.) How high does the rock go?

The rock will keep climbing and will eventually stop momentarily before falling down. At the point at which the velocity is zero the rock will reach its maximum height. So we first need to find the velocity formula: $v(t) = 160 - 32t$. Next we will set it equal to zero: $0 = 160 - 32t$. Solving for t gives us 5 seconds. Now we need to find the maximum height by plugging 5 back into the position equation: $s(5) = 160(5) - 16(5)^2 = 400$ ft.

b.) What is the velocity of the rock when it is 256 ft above the ground on the way up? On the way down?

So we want to find the times that correspond to a height of 256 ft. This means we will put in a 256 for s and solve for t:

$$256 = 160t - 16t^2$$
$$16t^2 - 160t + 256 = 0$$
$$16\left(t^2 - 10t + 16\right) = 0$$
$$16(t-2)(t-8) = 0$$

So t = 2 and 8 seconds. This means the rock is 256 ft above the ground at 2 seconds and at 8 seconds. Now we will find the corresponding heights:

$v(2) = 160 - 32(2) = 96$ ft/sec. The velocity is positive so it is moving up.
$v(8) = 160 - 32(8) = -96$ ft/sec. The velocity is negative so it is moving down.

c.) What is the acceleration of the rock at any time *t* during its flight (after the blast)?

We need to find the acceleration which is the derivative of our velocity function: $a(t) = -32 \ \ ft/\sec^2$. This means that acceleration is always constant. This is actually the acceleration due to gravity. As the rock rises it slows down. As it falls it speeds up.

d.) When does the rock hit the ground again?

When the object hits the ground the height is zero. Therefore set our position function $s(t) = 160t - 16t^2$ equal to zero: $0 = 160t - 16t^2$. So $0 = 16t(10 - t)$. Solving for *t* gives us $t = 0$ and 10 seconds. It is asking us when the object hits the ground again, so this will be at 10 seconds. So the object is in the air a total of 10 seconds.