# 3.5 Derivatives of Trigonometric Functions

**Derivatives of sine and cosine**

EXAMPLE: Let $f(x) = \sin(x)$. Find $f'(x)$ by using the limit process.

This is going back to what we learned last section. We need to use the limit form of the derivative:

$\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$. We need to find $f(x+h)$, which is $\sin(x+h)$. Now substitute in the formula.

$\lim\limits_{h\to 0} \dfrac{\sin(x+h) - \sin x}{h}$ We need to use the sum formula to break down $\sin(x+h)$.

$\lim\limits_{h\to 0} \dfrac{(\sin x \cos(h) + \cos x \sin(h)) - \sin x}{h}$ Everything in parenthesis is $\sin(x+h)$.

$\lim\limits_{h\to 0} \dfrac{\cos x \sin(h) + \sin x \cos(h) - \sin x}{h}$ I just switched the first two terms in order to do the next step.

$\lim\limits_{h\to 0} \dfrac{\cos x \sin(h) + \sin x(\cos(h) - 1)}{h}$ I factored out a sine from the last two terms.

$\lim\limits_{h\to 0} \dfrac{\cos x \sin(h)}{h} + \dfrac{\sin x(\cos(h) - 1)}{h}$ I divided each term by h.

What I see here are my special limits, which are $\lim\limits_{h\to 0} \dfrac{\sin(h)}{h} = 1$ and $\lim\limits_{h\to 0} \dfrac{1-\cos(h)}{(h)} = 0$. The first term has one of these special limits, but I need to factor out a negative from the numerator of the second fraction so that I can reverse the order of cosine and negative one.

$\lim\limits_{h\to 0} \dfrac{\cos x \sin(h)}{h} + \dfrac{-\sin x(-\cos(h) + 1)}{h}$ Now I will simplify.

$\lim\limits_{h\to 0} \dfrac{\cos x \sin(h)}{h} - \dfrac{\sin x(1 - \cos(h))}{h}$ Now I will break this up into different limits

$\lim\limits_{h\to 0} \cos x \cdot \lim\limits_{h\to 0} \dfrac{\sin(h)}{h} - \lim\limits_{h\to 0} \sin x \cdot \lim\limits_{h\to 0} \dfrac{(1-\cos(h))}{(h)}$ Now I see special limits I can use.

$\lim\limits_{h\to 0} \cos x \cdot 1 - \lim\limits_{h\to 0} \sin x \cdot 0$ I used the special limits. Now I will just simplify.

$f'(x) = \cos x$.

This same process can be done for $f(x) = \cos(x)$. The results are below:

**Derivatives for sine and cosine**

$\dfrac{d}{dx}[\sin x] = \cos x$ and $\dfrac{d}{dx}[\cos x] = -\sin x$

EXAMPLE: Given $f(x) = 4x - 5\cos x$, find $f'(x)$.

The derivative of the 4*x* will be 4, and the derivative of the cosine is negative sine, so $f'(x) = 4 - (-5\sin x)$. After simplifying your answer will be $f'(x) = 4 + 5\sin x$

EXAMPLE: Given $f(\theta) = \pi\sin\theta + \frac{1}{3\sqrt{\theta}}$, find $f'(\theta)$.

Even though there is a $\pi$, you will still multiply it by the derivative of sine, which is cosine. Then you will need to change the $\frac{1}{3\sqrt{\theta}}$ into $\frac{1}{3}\theta^{-1/2}$. Now take the derivative: $f'(\theta) = \pi\cos\theta + \frac{1}{3}\cdot -\frac{1}{2}\theta^{-3/2}$. The final answer is: $f'(\theta) = \pi\cos\theta - \frac{1}{6\theta^{\frac{3}{2}}}$.

EXAMPLE: Given $f(\theta) = \theta^2\sin\theta - 3\cos\theta$, find $f'(\theta)$.

The first term requires us to use the product rule since there are two different functions multiplied together. For the second term, we will take the derivative of cosine and multiply it by -3:

$f(\theta) = \theta^2\cos\theta + \sin\theta \cdot 2\theta + 3\sin\theta$ Now simplify.

$f(\theta) = \theta^2\cos\theta + 2\theta\sin\theta + 3\sin\theta$ At this point you could factor the last two terms, but it's not necessary.

EXAMPLE: Given $p = \frac{\cos q}{q^3}$, find $\frac{dp}{dq}$.

You could do this one with the quotient rule, but if you rewrite it as $p = q^{-3}\cos q$ then the product rule can be applied. Here the f is $q^{-3}$ and the g is $\cos q$. Now use the product rule:

$\frac{dp}{dq} = q^{-3}(-\sin q) + \cos q(-3q^{-4})$ Simplify and get rid of the negative exponents.

$\frac{dp}{dq} = \frac{-\sin q}{q^3} - \frac{3\cos q}{q^4}$ Now we need common denominators.

$\frac{dp}{dq} = \left(\frac{q}{q}\right)\frac{-\sin q}{q^3} - \frac{3\cos q}{q^4}$ Multiply the first by x over x to get common denominators.

$\frac{dp}{dq} = \frac{-q\sin q}{q^4} - \frac{3\cos q}{q^4}$ Combine into one fraction to get: $\frac{dp}{dq} = \frac{-q\sin q - 3\cos q}{q^4}$

This is as far as we can go.

EXAMPLE: Use the quotient rule to find $\frac{dr}{d\theta}$ if $r = \frac{\cos\theta}{1+\sin\theta}$.

For this one your *f* is $\cos\theta$ and *g* is $1+\sin\theta$. Now we apply the quotient rule.

$$\frac{dr}{d\theta} = \frac{\overset{g}{(1+\sin\theta)}\overset{f'}{(-\sin\theta)} - \overset{f}{\cos\theta}\overset{g'}{(\cos\theta)}}{(1+\sin\theta)^2}$$ How we distribute.

$$\frac{dr}{d\theta} = \frac{-\sin\theta - \sin^2\theta - \cos^2\theta}{(1+\sin\theta)^2}$$ We can factor out a negative from the last two terms.

$$\frac{dr}{d\theta} = \frac{-\sin\theta - (\sin^2\theta + \cos^2\theta)}{(1+\sin\theta)^2}$$ This allows us to use the identity $\sin^2 x + \cos^2 x = 1$.

$$\frac{dr}{d\theta} = \frac{-\sin\theta - 1}{(1+\sin\theta)^2}$$ We can factor out a negative again from the top.

$$\frac{dr}{d\theta} = \frac{-(1+\sin\theta)}{(1+\sin\theta)^2}$$ Now we can cancel the term $1+\sin x$.

$$\frac{dr}{d\theta} = \frac{-1}{1+\sin\theta}$$ This is our final answer.

EXAMPLE: Given $y = \tan x$, find $y'$.

Up to this point we have not learned how to take the derivative of this one, so we will use what we know. I will first use and identity to change the problem into: $f(x) = \frac{\sin x}{\cos x}$. Now I will apply the quotient rule.

$$y' = \frac{\overset{g}{\cos x}\overset{f'}{(\cos x)} - \overset{f}{\sin x}\overset{g'}{(-\sin x)}}{(\cos x)^2}$$ Now we simplify.

$$y' = \frac{\cos^2 x + \sin^2 x}{(\cos x)^2}$$ Now we use the identity $\sin^2 x + \cos^2 x = 1$.

$$y' = \frac{1}{(\cos x)^2}$$ Now we use the identity $\sec^2 x = \frac{1}{(\cos x)^2}$

$$y' = \sec^2 x.$$ So the derivative of $\tan x$ is $y' = \sec^2 x$

Similar processes will allow you to find the derivatives of the other trig functions. Here's a summary:

**Derivatives of Trig Functions**

$\frac{d}{dx}[\tan x] = \sec^2 x$ $\frac{d}{dx}[\cot x] = -\csc^2 x$ $\frac{d}{dx}[\sec x] = \sec x \tan x$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$

EXAMPLE: Find $y'$ if $y = \frac{\sec x}{x}$.

Here we use will use the quotient rule but we could also bring the x up and do the product rule.

$$y = \frac{\overset{g}{x}(\overset{f'}{\sec x \tan x}) - \overset{f}{\sec x}(\overset{g'}{1})}{x^2}$$

$$y = \frac{\sec x(x \tan x - 1)}{x^2}$$ I factored out a $\sec x$. This is as far as we can go.

EXAMPLE: Given $M(x) = (\cos x + 2\sin x)\csc x$, find $M'(x)$.

You could use the product rule here, however it will be easier to first multiply:
$M(x) = \cos x \cdot \csc x + 2\sin x \cdot \csc x$. It may look more complicated in this form, however we can apply trig identities: $M'(x) = \cos x \frac{1}{\sin x} + 2\sin x \frac{1}{\sin x}$. This can be simplified into $M'(x) = \cot x + 2$. Now we take the derivative of each term separately to get: $M'(x) = -\csc^2 x$.

EXAMPLE: Find $y''$ given $y = -3\sec x$. Write your answer in terms of secant only.

To begin, find the first derivative. We can find this directly by using the definition.

$y' = -3\sec x \tan x$ Now take the derivative of this to find the second derivative. We need to use a product rule here since we have multiplication of two functions. Here *f* = -3sec(*x*) and *g* = tan(*x*).

$y'' = -3\sec x \sec^2 x + \tan x \cdot -3\sec x \tan x$ We can simplify this.

$y'' = -3\sec^3 x - 3\sec x \tan^2 x$ Now factor out a -3sec(*x*).

$y'' = -3\sec x(\sec^2 x + \tan^2 x)$ Now use the identity $\tan^2 x = \sec^2 x - 1$

$y'' = -3\sec x(\sec^2 x + \sec^2 x - 1)$ Simplify

$y'' = -3\sec x(2\sec^2 x - 1)$ This is our final answer, expressed with all secants.

EXAMPLE: Determine the point(s) at which $y = x + \cos x$ has a horizontal tangent line on $[0, 2\pi]$.

First we take the derivative: $y' = 1 - \sin x$. Now set it equal to zero: $0 = 1 - \sin x$. We solve for sine: $\sin x = 1$. Now we look at a unit circle and find any angles whose y values are 1. You will find that $x = \frac{\pi}{2}$. This is the only place on the unit circle where the y value is 1.