3.7 Implicit Differentiation

Before I tell you what implicit differentiation is, let's start with an example:

EXAMPLE: Find $\frac{dy}{dx}$ if $xy + y = 3$.

This question is asking us to find the derivative of y with respect to x. In other words, x has to be the only variable. We can definitely solve for y. To do this, factor out a y. You will get $y(x+1) = 3$. Then divide to get: $y = \frac{3}{x+1}$. To get the derivative of this one we can first rewrite this as: $y = 3(x+1)^{-1}$. Now we can apply the power rule to get: $\frac{dy}{dx} = -3(x+1)^{-2}(1)$. This can be rewritten as: $\frac{dy}{dx} = \frac{-3}{(x+1)^2}$.

EXAMPLE: Find $\frac{dy}{dx}$ if $x^3 - 2y^2 + y = 4$.

The problem with this one is that if we try and solve for y we won't end up with an expression that has only x in it. Therefore we must do this by **implicit differentiation**. We will take the derivative of both sides of this equation without trying to solve for y. Whenever I take the derivative of an x term, the power rule can be applied as always. What about the y terms? Since we don't know what y is whenever we get to a part where we need to take the derivative of a y, we will get $\frac{dy}{dx}$. For a term like $2y^2$, when we take the derivative we will be applying the chain rule. The outside function can be done by using the power rule. The inside function is y, so again the derivative of this will be $\frac{dy}{dx}$. The derivative of $2y^2$ with respect to x will be $4y^1\frac{dy}{dx}$. Here is what the whole thing will look like once I take the derivative of both sides:

$3x^2 - 4y \cdot \frac{dy}{dx} + \frac{dy}{dx} = 0$. Notice there was a single y term next to the equals sign. The derivative of y is $\frac{dy}{dx}$. You want to now solve for $\frac{dy}{dx}$ since that is what the question is asking us to find.

$-4y \cdot \frac{dy}{dx} + \frac{dy}{dx} = -3x^2$ All the terms that have $\frac{dy}{dx}$ will be left on one side of the equation.

$\frac{dy}{dx}(-4y+1) = -3x^2$ We factored out the $\frac{dy}{dx}$. Now divide both sides by -4y + 1, which is also 1 – 4y.

$\frac{dy}{dx} = \frac{-3x^2}{1-4y}$ This is our answer. Usually you will have both x and y in your answer.

Now that we have seen the process of implicit differentiation, let's redo the first example we did only now we will use implicit differentiation.

EXAMPLE: Find $\frac{dy}{dx}$ if $xy + y = 3$.

This time we will not solve for y. To do implicit we must again take the derivative of both sides of the equation. On left side of the equation we have xy. This is not a single term. It is two terms multiplied together. For this one we must use the product rule. When we get to part of the product rule where we take the derivative of y we will always get $\frac{dy}{dx}$. Here is what the problem looks like after taking the derivative of both sides:

$f \quad g' \quad g \quad f'$

$x \cdot \frac{dy}{dx} + y(1) + \frac{dy}{dx} = 0$ Now we need to get all terms with $\frac{dy}{dx}$ on one side of the equation.

$x \cdot \frac{dy}{dx} + \frac{dy}{dx} = -y$ Now factor out the $\frac{dy}{dx}$.

$\frac{dy}{dx}(x+1) = -y$ Now divide both sides by x + 1 to get the answer: $\frac{dy}{dx} = \frac{-y}{x+1}$.

You will notice this is not the same answer as our example. To get that answer let's look at our first example. in that problem we got $y = \frac{3}{x+1}$. If we put this in for y we will get: $\frac{dy}{dx} = \frac{-\frac{3}{x+1}}{x+1}$. After simplifying we will get: $\frac{dy}{dx} = \frac{-3}{(x+1)^2}$ which is the same answer as in the first example. For all the problems in this section we will not know what y is so we don't need to substitute in anything for y. I just did it here to show you that implicit will give you the correct formula for the derivative.

EXAMPLE: Use implicit differentiation to find $\frac{dy}{dx}$ if $x^2y + y^2x = -2$.

When we take the derivative of both sides we need to use the product rule twice on the left hand side because there are two separate terms where x and y are multiplied together. When you use the product rule, when it gets to the part where we just write g, we are not taking the derivative of y, so that is why we don't put a $\frac{dy}{dx}$ in this term. This happens with the $y \cdot 2x$ term below: The only time you have a $\frac{dy}{dx}$ is when you actually take the derivative of y.

$f \quad g' \qquad g \quad f' \qquad f \quad g' \qquad g \qquad f'$

$x^2 \cdot \frac{dy}{dx} + y \cdot 2x + y^2(1) + x \cdot 2y\frac{dy}{dx} = 0$ Get all the $\frac{dy}{dx}$ terms on one side of the equation.

$x^2 \cdot \frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$ Now factor out the $\frac{dy}{dx}$.

$\frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2$ Divide both sides by $x^2 + 2xy$.

$\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$ Last step is to factor.

$\frac{dy}{dx} = \frac{-y(2x + y)}{x(x + 2y)}$ This is our answer.

EXAMPLE: Use implicit differentiation to find $\frac{dy}{dx}$ if $(2xy + 3)^2 = \sin y$.

We will take the derivative of both sides. On the left side the chain rule is applied. First the power rule is used to bring down 2. Then I did the derivative of the inside: $2(2xy+3)\left(2x \cdot \frac{dy}{dx} + y \cdot (1) + 0\right) = \cos y \frac{dy}{dx}$

Note that for the derivative of $2xy$, I needed to apply the product rule. Note the derivative of y is $\frac{dy}{dx}$

$$\overset{f}{2x} \cdot \overset{g'}{\frac{dy}{dx}} + \overset{g}{y} \cdot \overset{f'}{(2)}$$

On the right side I need to take the derivative of $\sin y$. This one also involves the chain rule. First I took the derivative of sine, which is cosine. But then I needed to take the derivative of y, which is where the $\frac{dy}{dx}$ came from. After simplifying the original result, we get: $(4xy + 6)\left(2x\frac{dy}{dx} + 2y\right) = \cos y \frac{dy}{dx}$. We need to multiply the two parenthesis together:

$8x^2y\frac{dy}{dx} + 8xy^2 + 12x\frac{dy}{dx} + 12y = \cos y\frac{dy}{dx}$. Now we will get all the $\frac{dy}{dx}$ terms on one side of the equation

$8x^2y\frac{dy}{dx} + 12x\frac{dy}{dx} - \cos y\frac{dy}{dx} = -8xy^2 - 12y$. Factor out $\frac{dy}{dx}$.

$\frac{dy}{dx}\left(8x^2y + 12x - \cos y\right) = -8xy^2 - 12y$ Solve for $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{-8xy^2 - 12y}{8x^2y + 12x - \cos y}$ This is our answer.

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EXAMPLE: Solve for $\frac{dy}{dx}$ by using implicit differentiation: $e^{xy}+x^2-y^2=10$.

For the first term, $u=xy$ and $u'=x\frac{dy}{dx}+y(1)$. So we need to take the derivative of both sides:

$e^{xy}\left(x\frac{dy}{dx}+y\right)+2x-2y\frac{dy}{dx}=0$. We can multiply the first two terms: $xe^{xy}\frac{dy}{dx}+ye^{xy}+2x-2y\frac{dy}{dx}=0$. Now we need to get all the terms with $\frac{dy}{dx}$ on one side of the equation: $xe^{xy}\frac{dy}{dx}-2y\frac{dy}{dx}=-ye^{xy}-2x$. Now factor out a $\frac{dy}{dx}$ from the left hand side: $\frac{dy}{dx}\left(xe^{xy}-2y\right)=-ye^{xy}-2x$. Solve for $\frac{dy}{dx}$ to get: $\frac{dy}{dx}=\frac{-ye^{xy}-2x}{xe^{xy}-2y}$.

EXAMPLE: Use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x^3+y^3=8$.

$3x^2+3y^2\frac{dy}{dx}=0$ Here we used the power rule on y^3 to do the derivative of the outside function. Once again the inside function is y, so that is why there is an extra $\frac{dy}{dx}$ attached to that term.

$3y^2\frac{dy}{dx}=-3x^2$ You want all the terms with $\frac{dy}{dx}$ on one side of the equation.

$\frac{dy}{dx}=\frac{-3x^2}{3y^2}=\frac{-x^2}{y^2}$ By dividing both sides by $3y^2$ we get $\frac{dy}{dx}=\frac{-x^2}{y^2}$.

Now we need to find the second derivative. We need to take the derivative again and use the quotient rule:

$\frac{d^2y}{dx^2}=\frac{y^2(-2x)-(-x^2)2y\frac{dy}{dx}}{y^4}$ I will simplify this expression.

$\frac{d^2y}{dx^2}=\frac{-2xy^2+2x^2y\frac{dy}{dx}}{y^4}$ I will divide both things on top by y^4.

$\frac{d^2y}{dx^2}=\frac{-2x}{y^2}+\frac{2x^2}{y^3}\cdot\frac{dy}{dx}$ We will substitute $\frac{-x^2}{y^2}$ for $\frac{dy}{dx}$ since we found this earlier.

$\frac{d^2y}{dx^2}=\frac{-2x}{y^2}+\frac{2x^2}{y^3}\cdot\left(\frac{-x^2}{y^2}\right)$ Simplify.

$\frac{d^2y}{dx^2}=\frac{-2x}{y^2}-\frac{2x^4}{y^5}$. Now we can get common denominators.

$\frac{d^2y}{dx^2}=\frac{-2xy^3-2x^4}{y^5}$ This is our final answer for the second derivative.

EXAMPLE: Verify that (2, 1) is on the curve $x^3 + y^3 = 4xy + 1$. Then find lines that are **(a)** tangent and **(b)** normal to the curve at (2, 1).

To verify if the point is on the curve, plug in a 2 for x and a 1 for y: $2^3 + 1^3 = 4(2)(1) + 1$. If you simplify both sides you will get $9 = 9$. Since both sides are the same it verifies that (2, 1) is on the curve. Now for part **(a)**.

(a) We must first find the derivative by implicit differentiation. On the right hand side we will use the product rule:

$$3x^2 + 3y^2 \frac{dy}{dx} = \overset{f}{4x} \cdot \overset{g'}{\frac{dy}{dx}} + \overset{g}{y} \cdot \overset{f'}{4}$$ We want all the terms with $\frac{dy}{dx}$ on one side of the equation.

$$3y^2 \frac{dy}{dx} - 4x \cdot \frac{dy}{dx} = 4y - 3x^2$$ Now factor out the $\frac{dy}{dx}$.

$$\frac{dy}{dx}(3y^2 - 4x) = 4y - 3x^2$$ Now divide both sides by $3y^2 - 4x$.

$$\frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}$$ This is our derivative. Now put in our point. So $x = 2$ and $y = 1$.

$$\frac{dy}{dx} = \frac{4(1) - 3(2)^2}{3(1)^2 - 4(2)} = \frac{-8}{-5} = \frac{8}{5}$$ So we now know our slope, m, is $\frac{8}{5}$. We will use y = mx + b to find b.

$1 = \frac{8}{5}(2) + b$, and after solving for b we get $b = \frac{-11}{5}$. The line $y = \frac{8}{5}x - \frac{11}{5}$ is tangent to the curve at (2, 1).

(b) If a line is normal to the curve, this means you want a line that is perpendicular to the tangent line but still passes through (2, 1). The line that is perpendicular will have a slope of $-5/8$. We will use y = mx + b to find b: $1 = -\frac{5}{8}(2) + b$, and after solving for b we get $b = 9/4$. Our equation is $y = -(5/8)x + 9/4$.

EXAMPLE: Verify that $(1,\ 0)$ is on the curve $2\cos(\pi x - y) - y = -2x$. Then find lines that are **(a)** tangent and **(b)** normal to the curve at $(1,\ 0)$.

To verify if the point is on the curve, plug in a 1 for x and 0 for y: $2\cos(\pi(1) - 0) - 0 = -2(1)$. If you simplify both sides you will get $2\cos\pi = -2$. So $2(-1) = -2$. Since both sides are the same it verifies that $(0,\ 1)$ is on the curve.

(a) We must first find the derivative by implicit differentiation. On the left hand side we will use the chain rule. The derivative of the outside will give us $-2\sin(\pi x - y)$. Then we need to multiply this by the derivative of the inside, which is $\pi x - y$. The derivative is $\pi - \frac{dy}{dx}$. After the cosine term we had a single y so its derivative is just $\frac{dy}{dx}$. After the equals sign we have a –2x, so its derivative is –2. Putting this all together gives us:

$-2\sin(\pi x - y)\left(\pi - \frac{dy}{dx}\right) - \frac{dy}{dx} = -2$ First distribute to get individual terms.

$-2\pi\sin(\pi x - y) - 2\sin(\pi x - y)\frac{dy}{dx} - \frac{dy}{dx} = -2$ We will keep terms with $\frac{dy}{dx}$ on one side of equation.

$-2\sin(\pi x - y)\frac{dy}{dx} - \frac{dy}{dx} = 2\pi\sin(\pi x - y) - 2$ Now factor out the $\frac{dy}{dx}$.

$\frac{dy}{dx}\left(-2\sin(\pi x - y) - 1\right) = 2\pi\sin(\pi x - y) - 2$ Divide both sides by what is after the $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{2\pi\sin(\pi x - y) - 2}{-2\sin(\pi x - y) - 1}$ Now we can plug in our point $(0,\ 1)$.

$\frac{dy}{dx} = \frac{2\pi\sin(\pi(1) - 0) - 2}{-2\sin(\pi(1) - 0) - 1} = \frac{2\sin\pi - 2}{-2\sin\pi - 1} = 2$ So we know that $m = \frac{dy}{dx} = 2$ since $\sin\pi = 0$.

We will use y = mx + b to find b. $0 = 2(1) + b$, and after solving for b we get $b = -2$. The line $y = 2x - 2$ is tangent to the curve at (1, 0).

(b) The line that is normal will have a slope of $-1/2$. We will use y = mx + b to find b. So $0 = -(1/2)(1) + b$, and after solving for b we get $b = 1/2$. The line normal to the curve at (1, 0) is $y = -(1/2)x + 1/2$.

EXAMPLE: You are given that $x \cdot \cos y = 1$. Find $\frac{dy}{dx}$ by implicit differentiation and evaluate the derivative at the given point, $\left(2, \frac{\pi}{3}\right)$.

The left side of the equation is a product, so we will first need to use the product rule.

$f \quad g' \quad g \quad f'$

$x \cdot (-\sin y) \cdot \frac{dy}{dx} + \cos y \cdot (1) = 0$ Now solve for $\frac{dy}{dx}$.

$-x\sin y \cdot \frac{dy}{dx} = -\cos y$ We want all the terms with $\frac{dy}{dx}$ to stay on one side of the equation.

$\frac{dy}{dx} = \frac{-\cos y}{-x\sin y}$ So we have that $\frac{dy}{dx} = \frac{\cos y}{x\sin y}$ which is the same as $\frac{dy}{dx} = \frac{\cot y}{x}$.

We now want to evaluate the derivative at $\left(2, \frac{\pi}{3}\right)$. To do this, plug in a 2 for x and a $\frac{\pi}{3}$ for y. You will get:

$$\frac{dy}{dx} = \frac{\cot\frac{\pi}{3}}{2} = \frac{1}{2}\cot\frac{\pi}{3} = \frac{1}{2} \cdot \frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{6}.$$