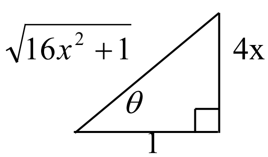
3.9 Derivatives of Inverse Trigonometric Functions

From trigonometry, we know that $\sin 30^\circ = \frac{1}{2}$. We put in an angle and get a value as a result. Recall that for inverse trig functions we put in the value and get an angle: $\sin^{-1}\frac{1}{2} = 30^\circ$. So here we put in the value of one half and got 30 degrees as a result. At the beginning of this course (Section 1.3) we did a brief review over inverse trigonometric functions. Now, let's use them for algebraic expressions.

EXAMPLE: Write in algebraic form: $\sec\left(\tan^{-1} 4x\right)$.

These problems involve drawing a triangle and labeling the sides with algebraic expressions. For all these problems we will assume that x is positive and the triangle should be drawn in the first quadrant. We can rewrite our problem as: $\sec\left(\tan^{-1}\frac{4x}{1}\right)$ We know that the adjacent side is 1 and the opposite side is 4x. We can use the Pythagorean theorem to find the hypotenuse: $c^2 = (4x)^2 + (1)^2$. So we have $c = \sqrt{16x^2+1}$

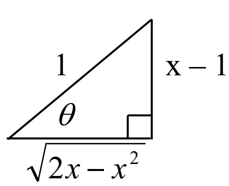


The secant on the outside of our problem tells us how to write our answer. From our drawing, secant is $\sqrt{16x^2+1}$ over 1 so we write our answer as:

$$\sec\left(\tan^{-1} 4x\right) = \sqrt{16x^2+1}$$

EXAMPLE: Write in algebraic form: $\cot\left(\sin^{-1}(x-1)\right)$.

These problems involve drawing a triangle and labeling the sides with algebraic expressions. For all these problems we will assume that x is positive and the triangle should be drawn in the first quadrant. We can rewrite our problem as: $\cot\left(\sin^{-1}\frac{x-1}{1}\right)$ We know that the opposite side is x – 1 and the hypotenuse is 1. We can use the Pythagorean theorem to find the hypotenuse: $1^2 = (x-1)^2 + a^2$. So we have $1 = x^2 - 2x + 1 + a^2$. Solving for a^2 we get $a^2 = 2x - x^2$, so $a = \sqrt{2x - x^2}$. Now we can draw the triangle.

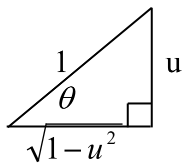


The cotangent on the outside of our problem tells us how to write our answer. From our drawing, cotangent is $\sqrt{2x-x^2}$ over x – 1 so we write our answer as:

$$\cot\left(\sin^{-1}(x-1)\right) = \frac{\sqrt{2x-x^2}}{x-1} \text{ where } x \neq 1.$$

Derivative of inverse trig functions

Let's find the derivative of $\sin^{-1} u$. In order to do this we will draw a triangle like in the previous problems. We can rewrite this problem as $\sin^{-1}\frac{u}{1}$. Then we know the hypotenuse is 1 and the opposite side is u. Then we can use the Pythagorean Theorem to find the remaining side. You will get this triangle:



On our problem, let's let $y = \sin^{-1} u$. The definition of the inverse sine tells us that $\sin y = u$. Now let's take the derivative of both sides with respect to x using implicit differentiation:

$\frac{d}{dx}[\sin y] = \frac{du}{dx}$ You will need to use the chain rule on the left side.

$\cos y \cdot y' = u'$ Now solve for y'

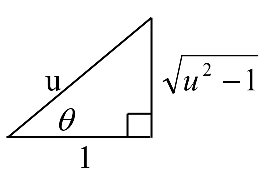
$y' = \frac{u'}{\cos y}$ From our triangle we know that $\cos y = \frac{\sqrt{1-u^2}}{1}$. Plug this in for $\cos y$.

$y' = \frac{u'}{\frac{\sqrt{1-u^2}}{1}}$ Now simplify and we have our answer.

$y' = \frac{u'}{\sqrt{1-u^2}}$ So if $y = \sin^{-1} u$ then $y' = \frac{u'}{\sqrt{1-u^2}}$.

EXAMPLE: If $y = \sec^{-1} u$, find y'.

In order to do this we will draw a triangle like in the previous problems. We can rewrite this problem as $\sec^{-1}\frac{u}{1}$. Then we know the hypotenuse is u and the adjacent side is 1. Then we can use the Pythagorean Theorem to find the remaining side. You will get this triangle:



The definition of the inverse secant tells us that $\sec y = u$. Now let's take the derivative of both sides with respect to x using implicit differentiation:

$\frac{d}{dx}[\sec y] = \frac{du}{dx}$ You will need to use the chain rule on the left side.

$\sec y \tan y \cdot y' = u'$ Now solve for y'

$y' = \frac{u'}{\sec y \tan y}$ We know that $\sec y = u$ and $\tan y = \sqrt{u^2 - 1}$ from our triangle. Now substitute.

$y' = \frac{u'}{u\sqrt{u^2 - 1}}$ So if $y = \sec^{-1} u$ then $y' = \frac{u'}{u\sqrt{u^2 - 1}}$.

Derivatives of Inverse Trig Functions

$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}\left[\cos^{-1} u\right] = -\frac{u'}{\sqrt{1-u^2}}$

$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{u'}{1+u^2}$ $\frac{d}{dx}\left[\cot^{-1} u\right] = -\frac{u'}{1+u^2}$

$\frac{d}{dx}\left[\sec^{-1} u\right] = \frac{u'}{|u|\sqrt{u^2-1}}$ $\frac{d}{dx}\left[\csc^{-1} u\right] = -\frac{u'}{|u|\sqrt{u^2-1}}$

EXAMPLE: Find the derivative: $y = \sec^{-1}(2x)$.

We will let $u = 2x$. Then $u' = 2$. We just need to substitute these into the formula $\frac{u'}{|u|\sqrt{u^2 - 1}}$. You will get:

$y' = \frac{2}{|2x|\sqrt{(2x)^2 - 1}}$. Now we can simplify: $y' = \frac{1}{|x|\sqrt{4x^2 - 1}}$. This is as far as we can go.

EXAMPLE: Find the derivative: $y = \cos^{-1}\left(5x^4\right)$.

We will let $u = 5x^4$. Then $u' = 20x^3$. We just need to substitute these into the formula $-\frac{u'}{\sqrt{1-u^2}}$. You will get: $y' = \frac{-20x^3}{\sqrt{1-\left(5x^4\right)^2}}$. Now we can simplify: $y' = \frac{-20x^3}{\sqrt{1-25x^8}}$. This is as far as we can go.

EXAMPLE: Find the derivative: $y = \cot^{-1}\left(3x^2 - 8\right)$.

We will let $u = 3x^2 - 8$. Then $u' = 6x$. We just need to substitute these into the formula $\frac{-u'}{1+u^2}$. You will get: $y' = \frac{-6x}{1+\left(3x^2+8\right)^2}$. Now we can simplify: $y' = \frac{-6x}{1+9x^4-48x^2+64}$. Our final answer is $y' = \frac{-6x}{9x^4-48x^2+65}$. This is as far as we can go.

EXAMPLE: Find the derivative: $y = \tan^{-1}\sqrt{5-2x}$.

We will let $u = \sqrt{5-2x}$. Then $u' = \frac{1}{2}(5-x)^{-\frac{1}{2}}(-2)$. This simplifies to $u' = -\frac{1}{\sqrt{5-2x}}$. Now we need to substitute these into the formula $\frac{u'}{1+u^2}$. You will get: $y' = -\frac{\frac{1}{\sqrt{5-2x}}}{1+\left(\sqrt{5-2x}\right)^2}$. Simplifying gives us:

$$y' = -\frac{\frac{1}{\sqrt{5-2x}}}{1+5-2x} = -\frac{\frac{1}{\sqrt{5-2x}}}{6-2x} = -\frac{1}{\sqrt{5-2x}} \cdot \frac{1}{6-2x}$$. So our final answer is: $y' = -\frac{1}{(6-2x)\sqrt{5-2x}}$.

EXAMPLE: Find the derivative: $y = \ln\left(\cot^{-1} 2x^3\right)$.

Since we have a natural log we will let $u = \cot^{-1} 4x^3$. Then $u' = -\frac{6x^2}{1+\left(2x^3\right)^2}$. This simplifies: $u' = -\frac{6x^2}{1+4x^6}$. To take the derivative of the *ln* we will use the formula $\frac{u'}{u}$. This will give us: $y = \frac{-\frac{6x^2}{1+4x^6}}{\cot^{-1} 4x^3}$. Now we can simplify: $y' = -\frac{6x^2}{\cot^{-1} 2x^3 \cdot \left(1+4x^6\right)}$. Nothing more we can do on this one.

EXAMPLE: Find the derivative: $y = x \cdot \tan^{-1}(2x) - \frac{1}{4}\ln(1+4x^2)$.

We need to use the product rule on this one. When we take the derivative of $\tan^{-1}(2x)$ we will let $u = 2x$. Then $u' = 2$ and we will use the formula the formula $\frac{u'}{1+u^2}$. For the second term since we have a natural log we will let $u = 1+4x^2$. Then $u' = 8x$. To do a derivative of the ln we will use the formula $\frac{u'}{u}$.

Using the product rule we get: $y' = x \cdot \frac{2}{1+(2x)^2} + \tan^{-1}(2x)\cdot(1) - \frac{1}{4}\cdot\frac{8x}{1+4x^2}$. Now we can simplify:

$y' = x \cdot \frac{2}{1+4x^2} + \tan^{-1}(2x) - \frac{2x}{1+4x^2}$. The first and last terms cancel and then we have: $y' = \tan^{-1}(2x)$.

EXAMPLE: Find the derivative: $y = 25\sin^{-1}\left(\frac{x}{5}\right) - x\sqrt{25-x^2}$. Write your answer as a single fraction.

For the first term we will let $u = \frac{x}{5}$. Then $u' = \frac{1}{5}$. You will use $\frac{u'}{\sqrt{1-u^2}}$. For the second term you will need to use the product rule combined with the chain rule. Putting this all together you will have:

$$y' = 25 \cdot \frac{\frac{1}{5}}{\sqrt{1-\left(\frac{x}{5}\right)^2}} - x \cdot \frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x) - \sqrt{25-x^2}\,(1)$$

Now we need to simplify.

$$y' = \frac{5}{\sqrt{1-\frac{x^2}{25}}} + \frac{x^2}{\sqrt{25-x^2}} + \sqrt{25-x^2}$$

For $1-\frac{x^2}{25}$ we can get common denominators: $\frac{25-x^2}{25}$.

$$y' = \frac{5}{\sqrt{\frac{25-x^2}{25}}} + \frac{x^2}{\sqrt{25-x^2}} + \sqrt{25-x^2}$$

This simplified further.

$$y' = \frac{5}{\frac{\sqrt{25-x^2}}{5}} + \frac{x^2}{\sqrt{25-x^2}} + \sqrt{25-x^2}$$

The first term can be simplified further.

$$y' = \frac{25}{\sqrt{25-x^2}} + \frac{x^2}{\sqrt{25-x^2}} + \frac{\sqrt{25-x^2}}{1}$$

Multiply the last term by $\frac{\sqrt{25-x^2}}{\sqrt{25-x^2}}$ to get common denominators.

$$y' = \frac{25+x^2-(25-x^2)}{\sqrt{25-x^2}}.$$

Now simplify the numerator and we are done.

$$y' = \frac{2x^2}{\sqrt{25-x^2}}.$$