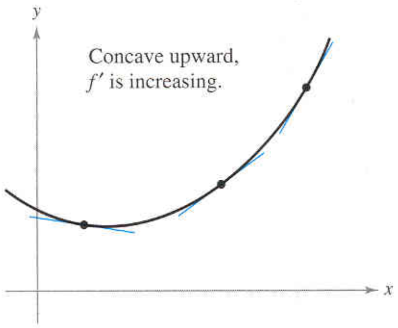
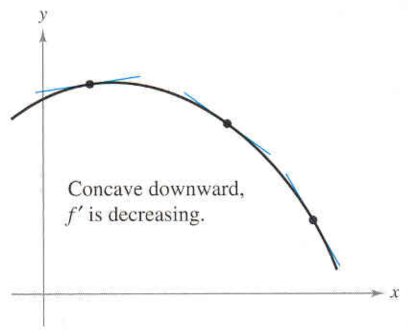
4.4 The Second Derivative Test and Curve Sketching

We can use the second derivative to tell us if a graph is concave up or concave down.





To see if something is concave down or concave up we need to look at the first derivative. If the first derivative is increasing, then it is concave up. If the first derivative is decreasing, then it is concave down. In order to tell if the derivative is increasing or decreasing, we need to look at the slope of the derivative, which is $f''(x)$.

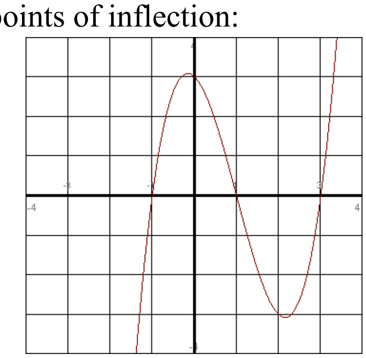
Second Derivative Test for Concavity

1.) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward in I.
2.) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward in I.

Hergert Number: Points where $f''(x) = 0$ or $f''(x)$ is undefined but are defined on $f(x)$.

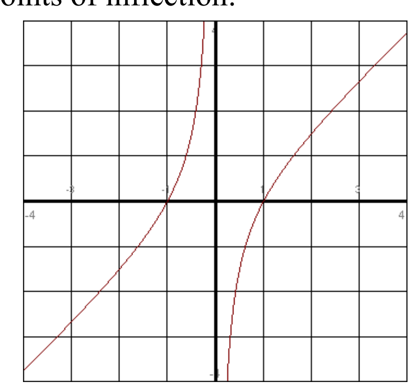
Inflection point – the point at which the concavity changes. To find the inflection point, first find the Hergert numbers. Then test to see if there is a sign change, which indicates a change in concavity.

EXAMPLE: Use the graph to indicate the intervals of concavity and any points of inflection:



We notice that the graph is concave down on $(-\infty,\ 1)$ since the graph opens down in this section. The graph is concave up on $(1,\ \infty)$ since the graph opens up in this section. The concavity changes at (1, 0), so this is our point of inflection.

EXAMPLE: Use the graph to indicate the intervals of concavity and any points of inflection:



This one is concave up on $(-\infty,\ 0)$ and concave down on $(0,\ \infty)$. There is no inflection point there seems to be a vertical asymptote at x = 0.

EXAMPLE: Given $f(x) = x^3(x-4)$ find all points of inflection and interval(s) of concavity.

First we need to find the second derivative. I will rewrite f as: $f(x) = x^4 - 4x^3$. The first derivative is $f'(x) = 4x^3 - 12x^2$. The second derivative is $f''(x) = 12x^2 - 24x$. We need to set the second derivative equal to zero: $0 = 12x^2 - 24x$. Factor to get: $0 = 12x(x-2)$. So x = 0 and 2. These are the Hergert numbers. This gives us our test intervals:

0 2

Now we need to pick test values just like you did in the last section. We need to test a value less than 0, in between 0 and 2, and we need to test a value greater than 2. I will use the test values -1, 1, and 3. **YOU NEED TO PUT THESE VALUES INTO THE <u>SECOND</u> DERIVATIVE,** $f''(x) = 12x^2 - 24x$ **!!!** If you put in a test value and you get a negative, then we will put a – on our table. If we get a positive, put a + sign in the table. You should get:

+	-	+

0 2

Any region that is positive will be concave up and any region with a negative will be concave down. Therefore the graph is concave down on $(0, 2)$ and concave up on $(-\infty, 0) \cup (2, \infty)$. Since the cavity does change, we know that x = 0 and x = 2 are inflection points. You want to find the y value associated with each x value we found, so put 0 and 2 into the ORIGINAL equation $f(x) = x^3(x-4)$ to get the y-values. The points of inflection are: (0, 0) and (2, -16).

EXAMPLE: Given $f(x) = 2x^4 - 8x + 3$ find all points of inflection and interval(s) of concavity.

The first derivative is $f'(x) = 8x^3 - 8$. The second derivative is $f''(x) = 24x^2$. Setting this equal to zero we get $0 = 24x^2$. Solving this we get the Hergert number, x = 0, which gives us our two test intervals on the table:

0

I will test x = -1 since it is less than 0 and I will test x = 1 since this is greater than 0. Again, put these test values into the SECOND DERIVATIVE function, $f''(x) = 24x^2$. The table will look like the following:

+	+

0

This graph is concave up on $(-\infty, 0) \cup (0, \infty)$. Since the concavity does not change, x = 0 is not a point of inflection, so this graph has no points of inflection. Therefore x = 0 is classified as a Hergert number, but not an inflection point.

More on next page.

EXAMPLE: Find all extrema, interval(s) of increasing/decreasing, critical points, interval(s) of concavity, points of inflection, intercepts, asymptotes and use this information to graph $y = -\frac{1}{3}x^3 + x - \frac{2}{3}$.

First we will find the first derivative to find any critical points: $y' = -x^2 + 1$. Setting this equal to zero we get $x = \pm 1$. We can set up the table and use test points with the FIRST DERIVATIVE:

-	+	-

-1 1

We see that the interval of increasing is (-1, 1) and the intervals of decreasing are $(-\infty, -1) \cup (1, \infty)$. The relative minimum is at $\left(-1, -\frac{4}{3}\right)$ and the relative maximum is at (1, 0). I got the y values for these points by using the ORIGINAL function.

The second derivative is $y'' = -2x$. Setting this equal to zero we will get the Hergert number, which is x = 0. We can put this on our table and use test points with the SECOND DERIVATIVE:

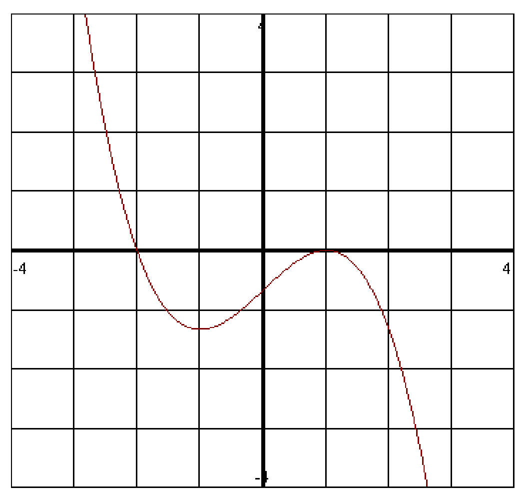
+	-

0

We see that the graph is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. The point of inflection is at $\left(0, -\frac{2}{3}\right)$. This is also our y-intercept.

We found that the relative maximum was at (1, 0). There is an x-intercept, but we need to see if there are any other intercepts. So we need to solve $0 = -\frac{1}{3}x^3 + x - \frac{2}{3}$. Multiplying both sides by -3 we get: $0 = x^3 - 3x + 2$. Now we can use synthetic division for precalculus:

1|	1	0	-3	2
		1	1	-2
	1	1	-2	0



So now we need to solve $x^2 + x - 2 = 0$. Factoring we get $(x-1)(x+2) = 0$, so x = 1 and x = -2. So now we found our second x-intercept which is at (-2, 0).

Now we are ready to graph our function. Since the graph goes to positive infinity as x gets smaller I know the graph will be coming down from the upper left part of my graph. It will pass through the points (-2, 0), $\left(-1, -\frac{4}{3}\right)$, $\left(0, -\frac{2}{3}\right)$, and (1, 0). Then it will go down and to the right since the graph goes to negative infinity as x gets large. We also know the graph will not pass through at (1, 0) since this is a relative max.

EXAMPLE: Find all extrema, interval(s) of increasing/decreasing, critical points, interval(s) of concavity, points of inflection, intercepts, asymptotes and use this information to graph $y = x(x-2)^3$.

First we will find the first derivative to find any critical points by using the product rule: $y' = x \cdot 3(x-2)^2(1) + (x-2)^3(1)$. We can factor out a common factor of $(x-2)^2$ to get $y' = (x-2)^2[3x + (x-2)]$. Simplifying we will get $y' = (x-2)^2(4x-2)$, or $y' = 2(x-2)^2(2x-1)$. Setting this equal to zero we get $x = \frac{1}{2},\ 2$. We can set up the table and use test points with the FIRST DERIVATIVE:

-	+	+

½ 2

We see that the interval of increasing is $\left(\frac{1}{2},\ 2\right) \cup (2,\ \infty)$and the intervals of decreasing are $\left(-\infty,\ \frac{1}{2}\right)$. The relative minimum is at $\left(\frac{1}{2},\ -\frac{27}{16}\right)$. There is no relative extrema at x = 2.

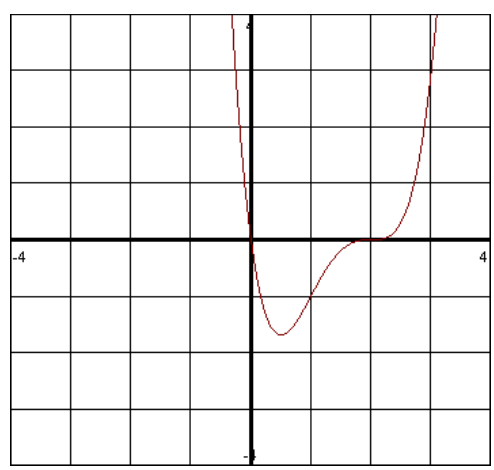
In order to get the second derivative you will need to use the product rule again. We will use $y' = 2(x-2)^2(2x-1)$ and take its derivative: $y'' = 2(x-2)^2(2) + (2x-1) \cdot 4(x-2)(1)$. After factoring out a common factor of $4(x-2)$ you will get: $y'' = 4(x-2)[(x-2) + (2x-1)]$ which simplifies to $y'' = 4(x-2)(3x-3)$ or $y'' = 12(x-2)(x-1)$. Setting this equal to zero we will get the Hergert numbers of x = 2. and x = 1. We can put these on our table and use test points with the SECOND DERIVATIVE:

+	-	+

1 2

We see that the graph is concave down on (1, 2) and concave up on $(-\infty,\ 1) \cup (2,\ \infty)$. The points of inflection are at $(1,\ -1)$ and (2, 0). We also know that (2, 0) is an x-intercept. The other one is at (0, 0). The y-intercept is also (0, 0).

Putting this all together we get the following graph. Notice that at (2, 0) there is a bend in the graph. This is not a relative max or min, however there is a point of inflection here.



EXAMPLE: Find all extrema, interval(s) of increasing/decreasing, critical points, interval(s) of concavity, points of inflection, intercepts, asymptotes and use this information to graph $y = x^{\frac{2}{3}}\left(x^2 - 4\right)$.

In this form we can find the intercepts. If we solve for the x-intercepts, we will put in a zero for y:
$0 = x^{\frac{2}{3}}\left(x^2 - 4\right)$ Solving this will give us x-intercepts (0, 0), (2, 0), and (–2, 0). This tells us that our y-intercept is also (0, 0). For the derivative, it is better to first multiply: $y = x^{\frac{8}{3}} - 4x^{\frac{2}{3}}$. Now we can take the first derivative: $y' = \frac{8}{3}x^{\frac{5}{3}} - \frac{8}{3}x^{-\frac{1}{3}}$. This simplifies to $y' = \frac{8}{3}x^{\frac{5}{3}} - \frac{8}{3x^{\frac{1}{3}}}$ Now we will set this to zero to find the critical points: $0 = \frac{8}{3}x^{\frac{5}{3}} - \frac{8}{3x^{\frac{1}{3}}}$. Now we move one term to the other side: $\frac{8x^{\frac{5}{3}}}{3} = \frac{8}{3x^{\frac{1}{3}}}$. Now cross multiply: $24x^2 = 24$ so we know $x = \pm 1$. The other critical point is where the first derivative is undefined. This will occur when $x = 0$. So now we put these in the table and use test points with the FIRST DERIVATIVE:

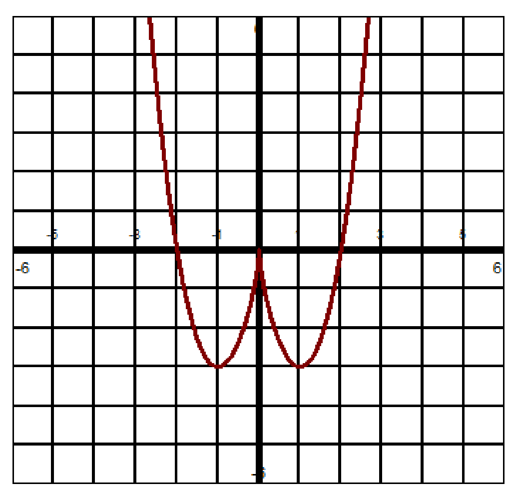
–	+	–	+
	–1	0	1

We see that the interval of decreasing is $(-\infty, -1) \cup (0, 1)$ and the intervals of increasing are $(-1, 0) \cup (1, \infty)$. The relative maximum is at (0, 0). The relative minimum is at $(-1, -3)$ and $(1, -3)$.

Now we will use the second derivative, so we will take the derivative of $y' = \frac{8}{3}x^{\frac{5}{3}} - \frac{8}{3}x^{-\frac{1}{3}}$. We will get:

$y'' = \frac{8}{3} \cdot \frac{5}{3}x^{\frac{2}{3}} - \frac{8}{3} \cdot -\frac{1}{3}x^{-\frac{4}{3}}$. This simplifies to: $y'' = \frac{40}{9}x^{\frac{2}{3}} + \frac{8}{9x^{\frac{4}{3}}}$. Now we will set this equal to zero

$0 = \frac{40}{9}x^{\frac{2}{3}} + \frac{8}{9x^{\frac{4}{3}}}$. We add one term to the other side: $\frac{40}{9}x^{\frac{2}{3}} = -\frac{8}{9x^{\frac{4}{3}}}$. Now cross multiply: $360x^2 = -72$. So $x^2 = -\frac{1}{5}$. There is no solution, so this does not produce a Hergert number. The other way to find Hergert numbers is to see where the second derivative is undefined. This occurs at $x = 0$. Therefore x=0 is the only Hergert number, and the only number on our table. After test points you will get:



+	+
	0

This graph is concave up on $(-\infty, 0) \cup (0, \infty)$ and not concave down anywhere. The table also tells us there are no points of inflection. Now let's put everything together and make our graph. Start by plotting the intercepts and the relative extrema. Then use the intervals of increasing, decreasing, and concavity to fill in the graph.

EXAMPLE: Given $y = x + \cos x$ find all points of inflection, critical points, interval(s) of concavity, interval(s) of increasing and interval(s) of decreasing on $[0,\ 2\pi]$. Then draw a sketch of this function.

The first derivative is $y' = 1 - \sin x$. We want to set this equal to zero to find the critical points: $0 = 1 - \sin x$. Solving this we will get: $\sin x = 1$ so $x = \dfrac{\pi}{2}$. This is the only critical point, so we will put it on the table and use test values 0 and π. Put these into the ORIGINAL FUNCTION $y = x + \cos x$. The table will be:

+	+

$\dfrac{\pi}{2}$

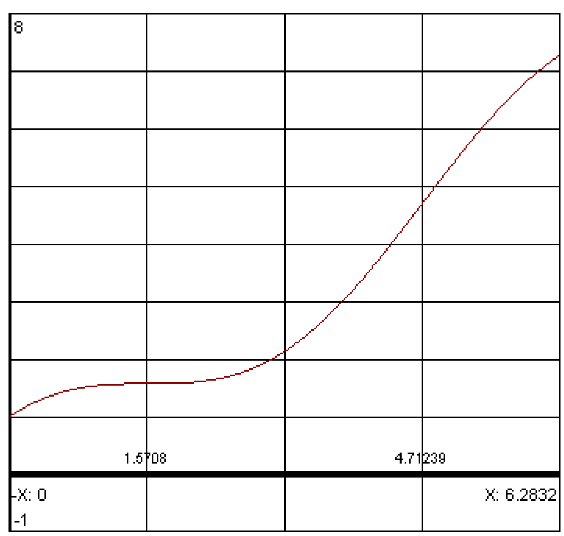
The intervals of increasing are $\left(0,\ \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2},\ 2\pi\right)$. There are no intervals of decreasing. Notice that I did not have infinity or negative infinity on this one since my interval is only on $[0,\ 2\pi]$. We see that there are no relative extrema since there is no change of sign when we did the first derivative test.

The second derivative is $y'' = -\cos x$. Setting this equal to zero you will get: $0 = -\cos x$. Divide both sides by -1 to get $0 = \cos x$. Solving this we will get the Hergert numbers of $x = \dfrac{\pi}{2},\ \dfrac{3\pi}{2}$. The table will only have these two values on it. Our test values will be $x = 0,\ \pi,\ 2\pi$. Remember to put these test values into the SECOND DERIVATIVE, $y'' = -\cos x$.

-	+	-

$\dfrac{\pi}{2}$ $\dfrac{3\pi}{2}$

The graph is concave down on $\left(0,\ \dfrac{\pi}{2}\right) \cup \left(\dfrac{3\pi}{2},\ 2\pi\right)$. The graph is concave up on $\left(\dfrac{\pi}{2},\ \dfrac{3\pi}{2}\right)$. The inflection points are $\left(\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right)$ and $\left(\dfrac{3\pi}{2},\ \dfrac{3\pi}{2}\right)$. We found the y-values of the points of inflection by putting the x-values into $y = x + \cos x$ (ORIGINAL). The decimal equivalents to these are: (1.57, 1.57) and (4.71, 4.71) For the graph, let's also find the values at the endpoints: $y = 0 + \cos 0$. This will give us one endpoint of (0, 1). For the other endpoint: $y = 2\pi + \cos(2\pi)$. This will give us the point $(2\pi,\ 1 + 2\pi)$ which is equivalent to (6.28, 7.28). For the graph, we start by plotting these points. Then we will use our concavity to figure out the curves. It is concave down from 0 to $\dfrac{\pi}{2}$ and then from $\dfrac{3\pi}{2}$ to 2π. Then it is concave up from $\dfrac{\pi}{2}$ to $\dfrac{3\pi}{2}$. Notice that this graph is increasing on the whole interval except at $\dfrac{\pi}{2}$. Here the slope is zero, so it is not increasing or decreasing.



EXAMPLE: Find all extrema, interval(s) of increasing/decreasing, critical points, interval(s) of concavity, points of inflection, intercepts, asymptotes and use this information to graph $y = \ln(5 - x^2)$.

We start by taking the first derivative by using the chain rule: $y' = -\frac{2x}{5 - x^2}$. We see that our derivative is undefined at $x = \pm\sqrt{5}$. However these will not be critical numbers because if we put them into the original function we will have *ln*(0) which is undefined. So next we need to set the first derivative equal to zero: $0 = -\frac{2x}{5 - x^2}$. Cross multiplying gives us $-2x = 0$, so $x = 0$. This is defined on the original so therefore this will be a critical number. Next we need to make a table. We need to first find the domain of the original function so that we will use the correct test numbers. To find the domain of the natural log we will set up this equation: $5 - x^2 > 0$ since we are only allowed numbers greater than zero for a normal logarithm. We know that we get a zero at $x = \pm\sqrt{5}$ so our domain will be $(-\sqrt{5}, \sqrt{5})$. So now we put 0 on the table in addition to our endpoints. We will use test points with the FIRST DERIVATIVE. You can use test points of 1 and –1. Below is the final sign configuration:

+	–

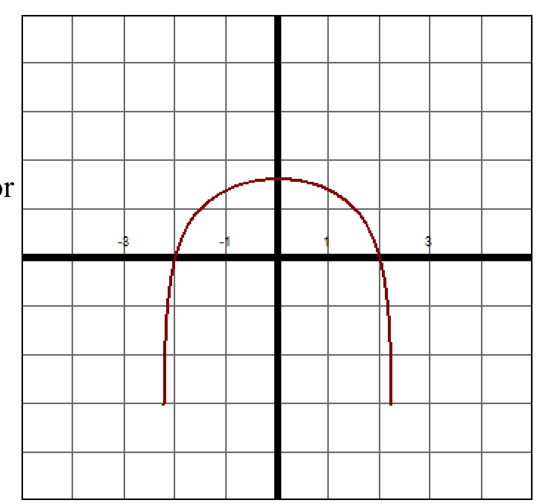
$-\sqrt{5}$ 0 $\sqrt{5}$

We see that the interval of increasing is $(-\sqrt{5}, 0)$ and the interval of decreasing is $(0, \sqrt{5})$. The relative maximum is at $(0,\ \ln 5)$ and as a decimal this is (0, 1.6). There is no relative minimum.

Now we will use the second derivative, so we will take the derivative of $y' = -\frac{2x}{5 - x^2}$. We will use the quotient rule. We will get: $y'' = \frac{(5 - x^2)(-2) - (-2x)(-2x)}{(5 - x^2)^2}$. Distributing will give you $y'' = \frac{-10 + 2x^2 - 4x^2}{(5 - x^2)^2}$ and this simplifies to $y'' = \frac{-2x^2 - 10}{(5 - x^2)^2}$. Now you can factor the top to get: $y'' = \frac{-2(x^2 + 5)}{(5 - x^2)^2}$. Now we will set this equal to zero $0 = \frac{-2(x^2 + 5)}{(5 - x^2)^2}$. Now cross multiply: $0 = -2(x^2 + 5)$. Solving this will yield imaginary solutions. The numbers that make the second derivative undefined will also make the original undefined $x = \pm\sqrt{5}$ so this means there are no Hergert numbers to put on our number line besides the endpoints from our domain. I will use zero as my test point and will use the SECOND DERIVATIVE:

–

$-\sqrt{5}$ $\sqrt{5}$



This graph is concave down on $(-\sqrt{5}, \sqrt{5})$ and there are no intervals for concave up. There are no inflection points. Let's find the x-intercepts since we already know our relative maximum is the y-intercept: $0 = \ln(5 - x^2)$. To solve we do this: $e^0 = e^{\ln(5 - x^2)}$. This this gives $1 = 5 - x^2$, so $x = \pm 2$. Plot this points in addition to the y-intercept. Then use our other information to sketch the graph. There should be vertical asymptotes at $x = \pm\sqrt{5}$. The lines on the graph to the right should keep going and will go down to negative infinity.

EXAMPLE: Find all extrema, interval(s) of increasing/decreasing, critical points, interval(s) of concavity, points of inflection, intercepts, asymptotes and use this information to graph $y = \dfrac{-64}{x^2 - 64}$.

First we need to note that the domain is $(-\infty, -8) \cup (-8, 8) \cup (8, \infty)$. We need to consider this when we find our critical values. The critical values must be defined on the original interval. Now let's find the derivative. We can rewrite this: $y = -64\left(x^2 - 64\right)^{-1}$. With it written this way we can use the chain rule:

$y' = 64\left(x^2 - 64\right)^{-2}(2x)$. This can be rewritten as: $y' = \dfrac{128x}{\left(x^2 - 64\right)^2}$. The derivative will be undefined at x = –8 and at x = 8. However these are not critical numbers since they are not on the domain of the original problem. Next we want to set the derivative equal to zero: $0 = \dfrac{128x}{\left(x^2 - 64\right)^2}$. After cross multiplying we will get $0 = 128x$.

Therefore the only critical number is at x=0. Next we want to create our sign chart for the first derivative. Even though 8 and –8 are not critical numbers, we still need to put these on our number line: Then we will pick test numbers on each interval. Remember to put these test numbers into the FIRST DERIVATIVE. The completed sign chart will look like this:

–	–	+	+
	–8	0	8

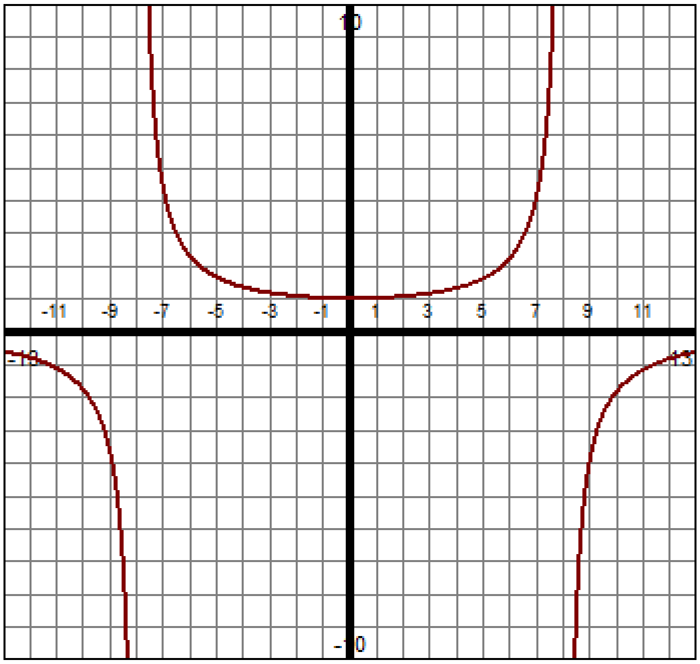
We see that the interval of decreasing is $(-\infty, 8) \cup (8, \infty)$ and the interval of increasing is $(0, 8) \cup (8, \infty)$. The relative minimum is at $(0,\ 1)$. There is no relative minimum.

Now we will use the second derivative, so we will take the derivative of $y' = \dfrac{128x}{\left(x^2 - 64\right)^2}$. We will use the quotient rule. We will get: $y'' = \dfrac{\left(x^2 - 64\right)^2(128) - (128x)\cdot 2\left(x^2 - 64\right)(2x)}{\left(\left(x^2 - 64\right)^2\right)^2}$. This simplifies to:

$y'' = \dfrac{128\left(x^2 - 64\right)^2 - (512x^2)\left(x^2 - 64\right)}{\left(x^2 - 64\right)^4}$. Now we can factor: $y'' = \dfrac{128\left(x^2 - 64\right)\left[\left(x^2 - 64\right) - 4x^2\right]}{\left(x^2 - 64\right)^4}$ which simplifies to $y'' = \dfrac{-128\left(3x^2 + 64\right)}{\left(x^2 - 64\right)^3}$ Now we will set this equal to zero $0 = \dfrac{-128\left(3x^2 + 64\right)}{\left(x^2 - 64\right)^3}$. Now cross multiply: $0 = -128\left(3x^2 + 64\right)$. Solving this will yield imaginary solutions. The numbers that make the second derivative undefined will also make the original undefined $x = \pm\sqrt{5}$ so this means there are no Hergert numbers to put on our number line besides the endpoints from our domain. We still need to put 8 and –8 on the number line since it makes the second derivative undefined. Put the test points into the SECOND DERIVATIVE:

–	+	–
	–8	8

The graph is concave down on $(-\infty, -8) \cup (8, \infty)$ and concave up on $(-8, 8)$. Now we can graph:



Note the vertical asymptotes at 8 and –8.

These kinds of rational functions also have horizontal asymptotes. To find them, we must find the limit:

$$\lim_{x\to\infty}\frac{-64}{x^2-64} \Rightarrow \lim_{x\to\infty}\frac{\frac{-64}{x^2}}{\frac{x^2}{x^2}-\frac{64}{x^2}} \Rightarrow \frac{0}{1-0}=0$$

Therefore, the horizontal asymptote is y = 0.

EXAMPLE: Sketch the curve $f(x)$ that meets the following conditions:

$f(-2)=f(2)=0$ and $f(0)=4$
$f'(-2)=f'(0)=f'(2)=0$
$f''(-1)=f''(0)=f''(1)=0$

Sign changes for $f'(x)$

–	+	–	+

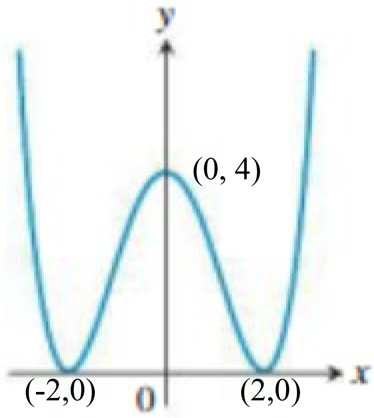
–2 0 2

Sign changes for $f''(x)$

+	–	+

–1 1

On the test, you will be asked to sketch a graph based on some given conditions. Let's start with the first set of information, $f(-2)=f(2)=0$ and $f(0)=4$. These are points that you can plot on the graph. It is saying that the graph crosses the x-axis and -2 and 2. The graph crosses the y-axis at 4. So you first want to plot these points. Now we will use the first derivative sign chart and the second derivative sign chart together to draw the graph. The graph is decreasing from negative infinity to x = -2 since the derivative is negative according to the sign chart. But at the same time that part of the graph is concave up. So we will draw a line that decreases but is also curving up. The graph is concave up until x = -1 and then it is concave down. The graph is increasing between x = -2 and 0. So between the x-values of -2 and -1 we need to draw a line that is concave up but is still increasing. Then between the x-values of -1 and 0 the graph will still be increasing, but concave down. Between the x-values of 0 and one we need to draw a line that is decreasing but also concave down. Then between the x-values of 1 and 2 the graph will be decreasing but also concave up. Finally, the line drawn after the x-value of 2 must be increasing and concave up. Your sketch should look something like the following:


y
(0, 4)
x
(-2,0)
0
(2,0)