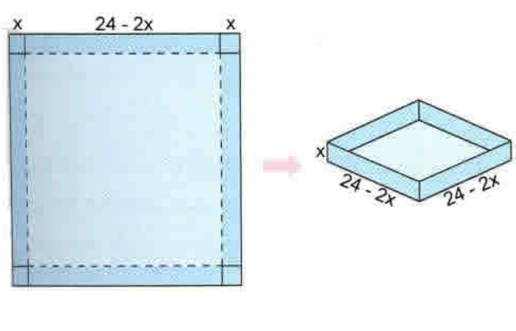
4.6 Applied Optimization Problems

In this section we will be looking at word problems where it asks us to maximize or minimize something. For all the problems in this section you will be taking the derivative of something and setting it equal to zero.

EXAMPLE: What is the smallest perimeter possible for a rectangle whose area is 36 square inches, and what are the dimensions?

For this one we can come up with two equations. We will use L for length and W for width. The first equation we have is $LW = 36$ since area equals length times width. The second equation is for the perimeter, and this is $2L + 2W = P$. We want to get one equation with a single variable, so we can take the first equation and solve for either L or W and put it into the second equation. I will solve the first equation for W: $W = \frac{36}{L}$. Now we can put this into the second equation to replace the W: $2L + 2\left(\frac{36}{L}\right) = P$. This is the same as $2L + 72L^{-1} = P$. Taking the derivative we will get: $P' = 2 - 72L^{-2}$. We need to set this equal to zero and rewrite this without negative exponents: $0 = 2 - \frac{72}{L^2}$. Add the second term to the other side to get: $\frac{72}{L^2} = 2$. Cross multiplying will give us $2L^2 = 72$. You will get $L^2 = 36$, so $L = \pm 6$. However we only want positive numbers, so we will just use L = 6. We can use the equation $W = \frac{36}{L}$ to find W. We will get $W = \frac{36}{6} = 6$. So L = 6 in and W = 6 in. To find the actual perimeter, just put these numbers back into $2L + 2W = P$. So $P = 2(6) + 2(6) = 24$ in.

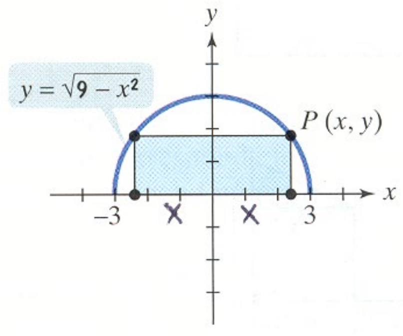
EXAMPLE: An open box of maximum volume is to be made from a square piece of material, 24 inches on a side, by cutting equal squares from the corners and turning up the sides (see figure). First write an equation for the volume, V, of the box as a function of x. Then find the maximum volume of the box.



To get the equation we notice that the height of this box is x. The length and width are both 24 – 2x. The formula for volume is V = LWH. So we have $V = (24-2x)(24-2x)(x)$, or $V = x(24-2x)^2$. We need to take the derivative and set it equal to zero. Using the product rule we will get: $V' = x \cdot 2(24-2x)(-2) + (24-2x)^2(1)$ We can factor out $24-2x$: $V' = (24-2x)(-4x + 24 - 2x)$ which is $V' = (24-2x)(24-6x)$. Setting this equal to zero you will get x = 4 and x = 12.

Even though we get two answers, the value x = 12 will not be included because 24 – 2(12) = 0 which means that a box will have a length or width of zero which is impossible. Therefore x = 4 is the only value for x that makes sense. To find the maximum volume, put this in the ORIGINAL equation for volume, which is $V = x(24-2x)^2$. You will get $V = 4(24-2(4))^2 = 1024$. Therefore the maximum area is 1024 square inches.

EXAMPLE: A rectangle is bounded by the x-axis and the semicircle $y=\sqrt{9-x^2}$ (see figure). What length and width should the rectangle have so that its area is a maximum? What is the maximum area?



First we need to find the area of the rectangle. The length is going to be 2x since there are two x's in the figure. The height of the box depends on where it hits the semicircle. The height will be equal to y, so $y=\sqrt{9-x^2}$. So now we can find the area formula: $A=2x\sqrt{9-x^2}$. We need to find this derivative and set it equal to zero. This will involve the product rule. First I will rewrite the problem as: $A=2x(9-x^2)^{\frac{1}{2}}$. Now take the derivative:

$A'=2x\cdot\frac{1}{2}(9-x^2)^{-\frac{1}{2}}(-2x)+(9-x^2)^{\frac{1}{2}}(2)$ Product and chain rules used here. Now simplify.

$A'=\frac{-2x^2}{\sqrt{9-x^2}}+2\sqrt{9-x^2}$ Now set the derivative equal to zero.

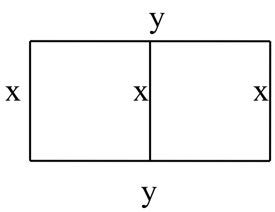
$0=\frac{-2x^2}{\sqrt{9-x^2}}+2\sqrt{9-x^2}$ Move one term to the left side.

$\frac{2x^2}{\sqrt{9-x^2}}=2\sqrt{9-x^2}$. Now cross multiply to get $2x^2=2(9-x^2)$. Distributing will give $2x^2=18-2x^2$. Now solve for x. You will have $4x^2=18$, so $x^2=\frac{9}{2}$. When you take the square root, just take the positive root. Then $x=\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{2}$. We know the length of the rectangle is 2x, so the length $=2\left(\frac{3\sqrt{2}}{2}\right)=3\sqrt{2}$. The height of the rectangle is $y=\sqrt{9-x^2}$. We will put our answer for x into this formula: $y=\sqrt{9-\left(\frac{3\sqrt{2}}{2}\right)^2}$. So $y=\sqrt{9-\frac{9}{2}}=\sqrt{\frac{9}{2}}=\frac{3\sqrt{2}}{2}$. The maximum area is these two answers multiplied together: $A=3\sqrt{2}\cdot\frac{3\sqrt{2}}{2}=9$.

EXAMPLE: Two sides of a triangle have lengths *a* and *b*, and the angle between them is θ. What value of θ will maximize the triangle's area? (Hint: $A=\frac{1}{2}ab\sin\theta$)

In this case a and b are to be treated as constants, or ordinary numbers. To find the maximum area we need to take the derivative and set it equal to zero. We will take the derivative with respect to θ. No product rule is needed here because the only variable is θ: $A'=\frac{1}{2}ab\cos\theta$. We now set this equal to zero: $0=\frac{1}{2}ab\cos\theta$. Now solve for cosine: $0=\cos\theta$. To solve for θ we need to take the inverse cosine of both sides: $\theta=\cos^{-1}(0)$. We will get $\theta=\frac{\pi}{2}$.

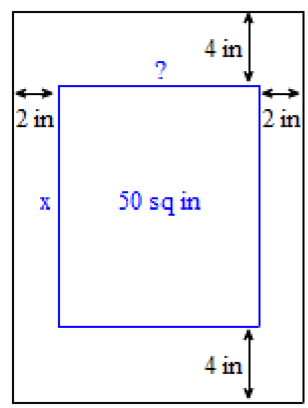
EXAMPLE: A rancher has 200 feet of fencing which to enclose two adjacent corrals. What dimensions should be used so that the enclosed area will be a maximum? (See figure)



In this problem I labeled the width x and the length y. It doesn't matter which variable you use. I can create two equations with this one. The first equation is $3x + 2y = 200$. This deals with the perimeter. The second equation is $A = xy$. You want to get one equation with a single variable, so we can solve for either x or y. Solving for y we get: $y = 100 - \frac{3x}{2}$. You want to put this into $A = xy$ for y to get $A = x\left(100 - \frac{3x}{2}\right)$ which is $A = 100x - \frac{3x^2}{2}$. The derivative is: $A' = 100 - 3x$. Setting it equal to zero we will get $0 = 100 - 3x$, so $x = \frac{100}{3}$. To find the y, we can use the equation $y = 100 - \frac{3x}{2}$ and put in $\frac{100}{3}$ for x. You will get: $y = 100 - \frac{3\left(\frac{100}{3}\right)}{2}$ which simplifies to y = 50. So the dimensions are $\frac{100}{3}$ by 50.

EXAMPLE: You are designing a poster to contain 50 square inches of printing with a 4 inch margin at the top and bottom and a 2 inch margin at each side. What overall dimensions will minimize the amount of paper used?

First we should draw a picture and label sides. I will make x be the height of the printed material and y to represent the ? in the picture below. From this picture we can make two equations.

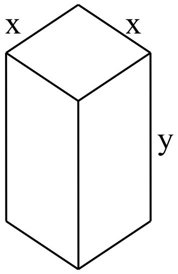


First, the area of the printed material is 50 square inches, so $50 = xy$. Another equation will involve the entire piece of paper including the margins. Here is another equation: $A = (x + 4)(y + 8)$. We need to solve the first equation for either x or y and substitute it into the second equation. I will solve for y in the first equation: $y = \frac{50}{x}$.

Now substitute into the second equation: $A = (x+4)\left(\frac{50}{x} + 8\right)$. Instead of applying the product rule it may be easier to distribute first: $A = 50 + 8x + \frac{200}{x} + 32$. This simplifies to $A = 82 + 8x + \frac{200}{x}$. It is now time to take the derivative. We will rewrite A as:

$A = 82 + 8x + 200x^{-1}$. The derivative is: $A' = 8 - 200x^{-2}$. Now set it equal to zero: $0 = 8 - \frac{200}{x^2}$. So $8 = \frac{200}{x^2}$. Cross multiply: $8x^2 = 200$. Divide to get: $x^2 = 25$, so $x = \pm 5$. We can't have a negative side, so $x = 5$ in. This means that $y = \frac{50}{5} = 10$ in. However the question asks us the find the overall dimensions, so the height will be 10 + 8 = 18 inches and the width is 5 + 4 = 9 inches. Therefore the overall dimensions of the paper are 9 inches by 18 inches.

EXAMPLE: A 108 cubic foot box with a square base and an open top is to be constructed from sheet metal of a given thickness. Find the dimensions of the tank with minimum weight.



A box with the minimum weight means we need to minimize the surface area. To find the surface area, we need to find the area of each individual side. There is one bottom that is x by x. Then there are four sides that have an area of x time y. So we have $S = x^2 + 4xy$. The volume of the box would be $V = x^2 y$. Since we are give the volume, we have the formula $108 = x^2 y$. We will solve this equation for y and substitute it into the other equation. $y = \frac{108}{x^2}$. Now put this into the surface area equation to get: $S = x^2 + 4x\left(\frac{108}{x^2}\right)$. This can be simplified to: $S = x^2 + 432x^{-1}$. Now we take the derivative: $S' = 2x - 432x^{-2}$. Setting this equal to zero we get: $0 = 2x - \frac{432}{x^2}$, so $\frac{432}{x^2} = 2x$. Clearing the fraction you will get $432 = 2x^3$, so $216 = x^3$ and x = 6. Then we can put 6 in for x in $y = \frac{108}{x^2}$ to get y = 3. So our dimensions are 3,6,6.

EXAMPLE: You have been asked to design a 2000 cubic centimeter can shaped like a right circular cylinder. What dimensions will use the least material. NOTE: The surface area formula for a right circular cylinder is $S = 2\pi r^2 + 2\pi rh$. The area of a right circular cylinder is $A = \pi r^2 h$.

We will ignore the thickness of the material and the waste in manufacturing. First we are given that the volume should be 2000 cubic centimeters, so one equation we will have is $2000 = \pi r^2 h$. We want to eliminate one variable in the surface area equation, so in our volume equation we can solve for either r or h. It will be easier to solve for h since that avoids square roots: $\frac{2000}{\pi r^2} = h$. Now we will substitute this into the surface area formula: $S = 2\pi r^2 + 2\pi r\left(\frac{2000}{\pi r^2}\right)$. This simplifies to $S = 2\pi r^2 + \frac{4000}{r}$. Because we want to minimize the amount of material used, this requires us to take the derivative and set it equal to zero. First we will rewrite S as: $S = 2\pi r^2 + 4000r^{-1}$. Now we take the derivative using the power rule: $S' = 4\pi r - 4000r^{-2}$. This can be rewritten as: $S' = 4\pi r - \frac{4000}{r^2}$. We now want to set it equal to zero and solve for r:

$0 = 4\pi r - \frac{4000}{r^2}$, $4\pi r = \frac{4000}{r^2}$, $4\pi r^3 = 4000$, $r^3 = \frac{1000}{\pi}$, so our answer for r is: $r = \frac{10}{\sqrt[3]{\pi}} \approx 6.82\ cm$. Then we also want to find h: $h = \frac{2000}{\pi r^2}$, so $h = \frac{2000}{\pi\left(\frac{10}{\sqrt[3]{\pi}}\right)^2}$. This simplifies to $h = \frac{20}{\pi^{\frac{1}{3}}} \approx 13.66\ cm$. So this means we have a special relationship. We see that the height needs to be twice the radius in order to use the least material.

EXAMPLE: The height above ground of an object moving vertically is given by $s = -16t^2 + 96t + 112$ with s in feet and t in seconds. Find a.) the object's velocity when t = 0, b.) its maximum height and when it occurs, and c.) its velocity when s = 0.

a.) The derivative of position will give velocity, so $s' = v = -32t + 96$. Now we put in a zero for t: $v = -32(0) + 96$. So the velocity is 96 ft/sec.

b.) We will take our velocity and set it equal to zero. This will occur when the object is at its maximum height: $0 = -32t + 96$. Solving for t will give us t = 3 seconds. Now we put this into the original equation for s: $s = -16(3)^2 + 96(3) + 112$. So s = 256 feet. This is the height.

c.) When s = 0 we have $0 = -16t^2 + 96t + 112$. Factoring this will give: $0 = -16(t^2 - 6t - 7)$, or $0 = -16(t+1)(t-7)$. Solving for t gives t = -1 and t = 7. We will ignore the negative time, so t = 7. Now that we know the time, we put this into the velocity formula: $v = -32(7) + 96$ = -128 ft/sec, which means it is falling back down.