5.1 Antiderivatives

Suppose we had $f(x)=x^3$ and we wanted to find the derivative. We can use the power rule: $f'(x)=3x^2$. What if we started with the derivative and we wanted to get back to the original function. This will involve the antiderivative.

Antiderivative: A function f is an antiderivative of f on an interval I if $f'(x)=f(x)$ for all x in I. The notation that is used for the antiderivative is the following: $\int$ You will put the function after this symbol. In our example we want to go from $f'(x)=3x^2$ back to the original, so the notation would look like this: $\int 3x^2dx$. The dx means that we are taking the derivative with respect to x, so x should be the only variable. To go from $f'(x)=3x^2$ to the original I would do the reverse steps of when I take the derivative. I need to add one to the power and divide by the new power: $f(x)=\frac{3x^{2+1}}{3}=x^3$. Well what if the original function was $f(x)=x^3+2$? It's derivative is still $f'(x)=3x^2$. How do we know what the original constant was? The answer is that we don't. In fact, we could have $f(x)=x^3+C$ and the derivative is still $f'(x)=3x^2$. So when we write our answer we write: $\int 3x^2dx=x^3+C$. You need the C as part of your answer. Here are some basic antiderivative formulas and properties:

Antiderivative formulas and properties:

$\int x^n\ dx=\frac{x^{n+1}}{n+1}+C$ Where $n\neq -1$, $\int 0\ dx=C$, $\int k\ dx=kx+C$

$\int k\cdot f(x)\ dx=k\int f(x)\ dx$, $\int f(x)+g(x)\ dx=\int f(x)\ dx+\int g(x)\ dx$

$\int \frac{1}{x}\ dx=\ln|x|+C$, $x\neq 0$, $\int e^{kx}dx=\frac{1}{k}e^{kx}+C$

$\int \cos kx\ dx=\frac{1}{k}\sin kx+C$, $\int \sin kx\ dx=-\frac{1}{k}\cos kx+C$

$\int \sec^2 kx\ dx=\frac{1}{k}\tan kx+C$, $\int \sec kx\cdot\tan kx\ dx=\frac{1}{k}\sec kx+C$

$\int \csc^2 kx\ dx=-\frac{1}{k}\cot kx+C$, $\int \csc kx\cdot\cot kx\ dx=-\frac{1}{k}\csc kx+C$

$\int \frac{1}{\sqrt{1-k^2x^2}}\ dx=\frac{1}{k}\sin^{-1}kx+C$, $\int \frac{1}{1+k^2x^2}\ dx=\frac{1}{k}\tan^{-1}kx+C$, $\int \frac{1}{x\sqrt{k^2x^2-1}}\ dx=\sec^{-1}kx+C$

$\int a^{kx}dx=\left(\frac{1}{k\ln a}\right)a^{kx}+C$, $a>0$, $a\neq 1$

EXAMPLE: Find the antiderivative for the function $4x^3$ when C equals 0.

To find this, we will use the formula $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$. We need to raise the power by 1 and divide by the new power. Then we will put in a 0 for C. We will get: $\frac{4x^{3+1}}{3+1} + 0 = x^4$.

EXAMPLE: Find the antiderivative for the function x^{-8} when C equals 0.

We will use the formula $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$. We need to raise the power by 1 and divide by the new power. Then we will put in a 0 for C. We will get: $\frac{x^{-8+1}}{-8+1} + 0 = \frac{x^{-7}}{-7}$ or $\frac{1}{-7x^7}$.

EXAMPLE: Find the antiderivative for the function $\sqrt[10]{x} + \frac{1}{\sqrt[10]{x}}$ when C equals 0.

First let's change the radicals into rational powers: $x^{\frac{1}{10}} + x^{-\frac{1}{10}}$ We will use the formula $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$. We need to raise the power by 1 and divide by the new power for each term. Then we will put in a 0 for C. We will get: or $\frac{x^{\frac{1}{10}+1}}{\frac{1}{10}+1} + \frac{x^{-\frac{1}{10}+1}}{-\frac{1}{10}+1} = \frac{x^{\frac{11}{10}}}{\frac{11}{10}} + \frac{x^{\frac{9}{10}}}{\frac{9}{10}} = \frac{10}{11}x^{\frac{11}{10}} + \frac{10}{9}x^{\frac{9}{10}}$.

EXAMPLE: Find the antiderivative for the function $\sin(12x) + \cos(12x)$ when C equals 0.

We will use the formula $\int \sin kx \, dx = -\frac{1}{k}\cos kx + C$ and $\int \cos kx \, dx = \frac{1}{k}\sin kx + C$ with a 0 for C. In each case, the *k*-value is 12. Using the formulas, we will get: $-\frac{1}{12}\cos(12x) + \frac{1}{12}\sin(12x)$.

EXAMPLE: Find the antiderivative for the function $-\csc^2\left(\frac{2x}{5}\right)$ when C equals 0.

We will use the formula $\int \csc^2 kx \, dx = -\frac{1}{k}\cot kx + C$ with a 0 for C. In this case, the *k*-value is $\frac{2}{5}$. Using the formula we will get: $-\left(-1\Big/\frac{2}{5}\right)\cot\left(\frac{2x}{5}\right) = \frac{5}{2}\cot\left(\frac{2x}{5}\right)$.

EXAMPLE: Find the antiderivative for the function $\dfrac{1}{x\sqrt{169x^2-1}}$ when C equals 0.

Looking at the antiderivative formulas, we need to find the one that matches this form. It will be one of the inverse trigonometric functions. We will use the formula $\int \dfrac{1}{x\sqrt{k^2x^2-1}}\,dx = \sec^{-1} kx + C$ with a 0 for C. In this case, the *k*-value is 13 since $169 = 13^2$. Using the formula we will get: $\sec^{-1}(13x)$.

EXAMPLE: Find the antiderivative for the function $e^{3x} - e^{\frac{x}{7}} - \dfrac{1}{x}$ when C equals 0.

The first two terms will use the formula $\int e^{kx}dx = \dfrac{1}{k}e^{kx} + C$ with a 0 for C. For the last term you will use $\int \dfrac{1}{x}\,dx = \ln|x| + C$. Using these formulas we will get $\dfrac{1}{3}e^{3x} - \dfrac{1}{\frac{1}{7}}e^{\frac{x}{7}} - \ln|x| \;=\; \dfrac{1}{3}e^{3x} - 7e^{\frac{x}{7}} - \ln|x|$

EXAMPLE: Find the indefinite integral: $\int (4x^3 + 6x^2 - 3)\,dx$ and check your answer.

We find the antiderivative using the formula: $\int (4x^3 + 6x^2 - 3)\,dx = \dfrac{4x^{3+1}}{3+1} + \dfrac{6x^{2+1}}{2+1} - \dfrac{3x^{0+1}}{0+1} + C$. This simplifies to: $\int (4x^3 + 6x^2 - 3)\,dx = x^4 + 2x^3 - 3x + C$.

We can check our answer by differentiation. Let $f(x) = x^4 + 2x^3 - 3x + C$. Then $f'(x) = 4x^3 + 6x^2 - 3$.

Thus $\int f'(x)\,dx = f(x) + C$.

EXAMPLE: Find the indefinite integral: $\int \dfrac{3}{x^2} - \dfrac{1}{5\sqrt{x}} + \dfrac{3}{4}dx$.

First we can rewrite this as: $\int 3x^{-2} - \dfrac{1}{5}x^{-\frac{1}{2}} + \dfrac{3}{4}\,dx$. Then we find the antiderivative using the property. We add 1 to the new power and divide by the new power for the first two terms. For the last term, since it is a constant we will only need to add an *x*: $3 \cdot \dfrac{x^{-2+1}}{-2+1} - \dfrac{1}{5} \cdot \dfrac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \dfrac{3}{4}x + C$. This simplifies to

$\dfrac{3x^{-1}}{-1} - \dfrac{1}{5} \cdot \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} + \dfrac{3}{4}x + C$ So our final answer is: $-\dfrac{3}{x} - \dfrac{2}{5}\sqrt{x} + \dfrac{3}{4}x + C$.

EXAMPLE: Find the indefinite integral: $\int x^{-3}(x+1)+\sin(3x)\ dx$

For this one, it is better to first distribute: $\int x^{-2}+x^{-3}+\sin(3x)\ dx$. Now we add 1 and divide by the new power for each term and follow our formula for the sine: $\frac{x^{-1}}{-1}+\frac{x^{-2}}{-2}-\frac{1}{3}\cos(3x)+C$. We can rewrite this as: $-\frac{1}{x}-\frac{1}{2x^2}-\frac{1}{3}\cos(3x)+C$.

EXAMPLE: Find the indefinite integral: $\int \frac{5}{e^{2x}}+3^{4x}\ dx$.

First we will rewrite it as: $\int 5e^{-2x}+3^{x}\ dx$. Now we will follow the formulas given for e and for a number raised to an x: $5\cdot -\frac{1}{2}e^{-2x}+\frac{1}{4\ln 3}\cdot 3^{4x}+C$. So our final answer is: $-\frac{5}{2e^{2x}}+\frac{3^{4x}}{4\ln 3}+C$ or $-\frac{5}{2e^{2x}}+\frac{81^{x}}{4\ln 3}+C$.

EXAMPLE: Find the indefinite integral: $\int\left(\frac{x\sqrt{x}+2\cdot\sqrt[3]{x}-x}{x^2}\right)dx$.

First we will rewrite using rational powers. Note for the first term we have $x^1\cdot x^{\frac{1}{2}}=x^{\frac{3}{2}}$: $\int\left(\frac{x^{\frac{3}{2}}+2x^{\frac{1}{3}}-x}{x^2}\right)dx$.

We can divide everything in the numerator by x^2 which involves taking the top power and subtracting the bottom power: $\int\left(x^{-\frac{1}{2}}+2x^{-\frac{5}{3}}-\frac{1}{x}\right)dx$. Now apply the antiderivative. Notice the last term turns into ln: $\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+2\cdot\frac{x^{-\frac{2}{3}}}{-\frac{2}{3}}-\ln|x|+C$. This simplifies to: $2\sqrt{x}-\frac{3}{x^{\frac{2}{3}}}-\ln|x|+C$.

EXAMPLE: Find the indefinite integral: $\int(\theta^2+\sec^2 7\theta)\ d\theta$.

We can apply the antiderivative formula directly: $\int(\theta^2+\sec^2 7\theta)\ d\theta=\frac{\theta^3}{3}+\frac{1}{7}\tan 7\theta+C$.

EXAMPLE: Find the indefinite integral: $\int \left(\frac{\cos(4\theta)}{1-\cos^2(4\theta)} \right) d\theta$.

We want to use some identities on this one. First we know that $1-\cos^2(4\theta) = \sin^2(4\theta)$. Then our problem becomes: $\int \left(\frac{\cos 4\theta}{\sin^2 4\theta} \right) d\theta$. This can be rewritten as: $\int \frac{1}{\sin 4\theta} \cdot \frac{\cos 4\theta}{\sin 4\theta} \, d\theta$ which is the same as $\int \csc 4\theta \cot 4\theta \, d\theta$. The antiderivative tells us $\int \csc 4\theta \cot 4\theta \, d\theta = -\frac{1}{4}\csc 4\theta + C$

EXAMPLE: Find the indefinite integral: $\int \frac{dx}{1+4x^2}$.

We need to recognize that this this will turn into an inverse trig function. We can rewrite this as $\int \frac{dx}{1+2^2x^2}$. Then we know that $k = 2$. We apply the formula to get the antiderivative: $\int \frac{1}{1+2^2x^2} \, dx = \frac{1}{2}\tan^{-1} 2x + C$.

EXAMPLE: Solve the initial value problem: Given: $\frac{dy}{dx} = 6x^2$ and $y(0) = -1$.

The definition tells us $\int \frac{dy}{dx} \, dx = y(x)$. So this means $\int 6x^2 \, dx = y(x)$. We will take the antiderivative and get: $y(x) = \frac{6x^3}{3} + C$, which is: $y(x) = 2x^3 + C$. In order to find the value for C we need to use the information they gave us: $y(0) = -1$. This means we put in a 0 for x. You will get: $y(0) = 2(0)^3 + C$. We know that $y(0) = -1$ so $-1 = 0 + C$, so we know that $C = -1$. So our equation is $y(x) = 2x^3 - 1$.

EXAMPLE: Solve the initial value problem: Given $\frac{d^2y}{dx^2} = x^2$ and $y'(0) = 6$ and $y(0) = 3$.

This is a two step problem. First we will find $y'(x)$ and then $y(x)$. To find the first derivative we need to do: $y'(x) = \int x^2 \, dx$. This results in: $y'(x) = \frac{x^3}{3} + C$. We know $y'(0) = 6$, so we can plug in a zero for x: $y'(0) = \frac{0^3}{3} + C$. Then we have $y'(0) = C$. So C = 6. So we have our first derivative: $y'(x) = \frac{x^3}{3} + 6$. Now we need to do $y(x) = \int \left(\frac{x^3}{3} + 6 \right) dx$. You will get: $y(x) = \frac{x^4}{12} + 6x + C$. Now we will use $y(0) = 3$. We will get $3 = \frac{0^4}{12} + 6(0) + C$, so we know that C = 3. Our equation is $y(x) = \frac{x^4}{12} + 6x + 3$.

EXAMPLE: Solve the initial value problem: Given $\frac{d^2r}{d\theta^2} = \sin 5\theta$ and $r'(0) = 1$ and $r(0) = 6$.

This is a two step problem. First we will find $r'(\theta)$ and then $r(\theta)$. To find the first derivative we need to do: $r'(\theta) = \int \sin 5\theta \; d\theta$. This results in: $r'(\theta) = -\frac{1}{5}\cos 5\theta + C$. We know $r'(0) = 1$, so we can plug this in: $1 = -\frac{1}{5}\cos(5 \cdot 0) + C$. Since cos(0) = 1 then we have $1 = -\frac{1}{5} + C$, so $C = \frac{6}{5}$. So we have our first derivative: $r'(\theta) = -\frac{1}{5}\cos 5\theta + \frac{6}{5}$. Now we need to do $r(\theta) = \int \left(-\frac{1}{5}\cos 5\theta + \frac{6}{5} \right) d\theta$. You will get: $r(\theta) = -\frac{1}{25}\sin 5\theta + \frac{6}{5}\theta + C$. Now we will use $r(0) = 6$. We will get $6 = -\frac{1}{25}\sin(5 \cdot 0) + \frac{6}{5}(0) + C$. Since sin(0) = 0 we have $6 = 0 + 0 + C$, so C = 6. Our equation is $r(\theta) = -\frac{1}{25}\sin 5\theta + \frac{6}{5}\theta + 6$.