5.3 Double Angle, Half Angle, and Reduction Formulas

If we have either a double angle $2\theta$ or a half angle $\frac{\theta}{2}$ then these have special formulas:

**Double Angle Formulas**

\[
\sin(2\theta) = 2\sin \theta \cos \theta
\]

\[
\cos(2\theta) = \cos^2 \theta - \sin^2 \theta
\]

\[
\cos(2\theta) = 2\cos^2 \theta - 1 \quad \text{There are three formulas for } \cos(2\theta)
\]

\[
\cos(2\theta) = 1 - 2\sin^2 \theta
\]

\[
\tan(2\theta) = \frac{2\tan \theta}{1 - \tan^2 \theta}
\]

The above double angle formulas can be manipulated to derive power reducing formulas. These formulas will be useful primarily in Calculus 2:

**Power Reducing Formulas**

\[
\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}
\]

\[
\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}
\]

\[
\tan^2 \theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}
\]

Besides double angle formulas there are also half angle formulas:

**Half Angle Formulas**

\[
\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} \quad \text{You will choose plus or minus depending on what quadrant } \frac{\theta}{2} \text{ is.}
\]

\[
\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}} \quad \text{You will choose plus or minus depending on what quadrant } \frac{\theta}{2} \text{ is.}
\]

\[
\tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \quad \text{You will choose plus or minus depending on what quadrant } \frac{\theta}{2} \text{ is.}
\]

There are better formulas for $\tan \frac{\theta}{2}$ that don’t involve a plus or minus:

\[
\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}, \quad \tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta}
\]
EXAMPLE: Compute $\sin \theta$, $\tan \theta$, $\csc \theta$, $\sec \theta$, $\cot \theta$, $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$, $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, and $\tan \frac{\theta}{2}$ if you are given $\cos \theta = -0.8$ and $90^\circ \leq \theta \leq 180^\circ$. Round decimals to two decimal places.

We need to draw a triangle for this one. We are given that the triangle should be drawn in the second quadrant. We can rewrite our problem as $\cos \theta = -\frac{0.8}{1}$. We know the adjacent side is $-0.8$. The hypotenuse is 1. Once we label our triangle we find the third side by using the Pythagorean theorem.

$$\theta$$

Now we can get our first 5 trigonometric functions by reading off our triangle:

$$\sin \theta = 0.6 \quad \csc \theta = \frac{1}{0.6} = 1.67 \quad \sec \theta = \frac{1}{-0.8} = -1.25$$

$$\tan \theta = \frac{0.6}{-0.8} = -0.75 \quad \cot \theta = \frac{-0.8}{0.6} = -1.33$$

Next I will find $\sin(2\theta)$ by using its formula: $\sin(2\theta) = 2 \sin \theta \cos \theta$. We already know sine and cosine, so we will substitute in those decimals: $\sin(2\theta) = 2(0.6)(-0.8) = -0.96$.

Next I will find $\cos(2\theta)$ by using its formula: $\cos(2\theta) = 2 \cos^2 \theta - 1$. Notice I had a choice of three formulas to use. Any of them would give you the correct answer. We already know $\cos \theta = -0.8$, so we will substitute in this decimal: $\cos(2\theta) = 2(-0.8)^2 - 1 = 0.28$.

Next I will find $\tan(2\theta)$ by using its formula: $\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$. We already know $\tan \theta = -0.75$, so we will substitute in this decimal: $\tan(2\theta) = \frac{2(-0.75)}{1 - (-0.75)^2} = \frac{-1.5}{0.4375} = -3.43$.

For the half angle formulas I need to determine which quadrant $\frac{\theta}{2}$ is in. To do this let’s first start with our given statement $90^\circ \leq \theta \leq 180^\circ$. If I divide everything by two we get: $45^\circ \leq \frac{\theta}{2} \leq 90^\circ$. This tells us that $\frac{\theta}{2}$ is in the first quadrant, so sine, cosine, and tangent of $\frac{\theta}{2}$ should all be positive.
Now I will find $\sin \frac{\theta}{2}$ by using its formula: 
$$\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}}.$$ 
We chose a positive because $\frac{\theta}{2}$ is in the second quadrant. We already know $\cos \theta = -0.8$, so we will substitute in this decimal:
$$\sin \frac{\theta}{2} = \sqrt{\frac{1-(-0.8)}{2}} = \sqrt{\frac{1.8}{2}} = \sqrt{0.9} = 0.95.$$

Now I will find $\cos \frac{\theta}{2}$ by using its formula: 
$$\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}}.$$ 
We chose a positive because $\frac{\theta}{2}$ is in the second quadrant. We already know $\cos \theta = -0.8$, so we will substitute in this decimal:
$$\cos \frac{\theta}{2} = \sqrt{\frac{1+(-0.8)}{2}} = \sqrt{\frac{0.2}{2}} = \sqrt{0.1} = 0.32.$$

Finally I will find $\tan \frac{\theta}{2}$. I have three formulas to choose. I will choose: 
$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}.$$ 
We already know $\cos \theta = -0.8$, and $\sin \theta = 0.6$ so we will substitute in these decimals: 
$$\tan \frac{\theta}{2} = \frac{0.6}{1+(-0.8)} = 3.$$

EXAMPLE: Compute $\sin \theta$, $\tan \theta$, $\sec \theta$, $\csc \theta$, $\cot \theta$, $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$, $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, and $\tan \frac{\theta}{2}$ if you are given $\cot \theta = \frac{1}{2}$ and $\sin \theta < 0$.

We are given than the cotangent is positive and sine is negative. This only occurs in the 3rd quadrant. We know the adjacent side is 1 and the opposite side is 2. However, because we are in the third quadrant we need to make the 1 and 2 negative. Once we label our triangle we find the third side by using the Pythagorean theorem. This will give us $\sqrt{5}$.

Now we can get our first 5 trigonometric functions by reading off our triangle:
$$\sin \theta = -\frac{2}{\sqrt{5}} = -2\sqrt{5}$$
$$\cos \theta = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$
$$\sec \theta = -\sqrt{5}$$
$$\csc \theta = -\frac{\sqrt{5}}{2}$$
$$\tan \theta = 2$$
Next I will find $\sin(2\theta)$ by using its formula: $\sin(2\theta) = 2\sin \theta \cos \theta$. We already know sine and cosine, so we will substitute in those fractions: 

$$\sin(2\theta) = 2\left(-\frac{2\sqrt{5}}{5}\right) \left(-\frac{\sqrt{5}}{5}\right) = \frac{20}{25} = \frac{4}{5}.$$ 

Next I will find $\cos(2\theta)$ by using its formula: $\cos(2\theta) = 2\cos^2 \theta - 1$. Notice I had a choice of three formulas to use. Any of them would give you the correct answer. We already know $\cos \theta = -\frac{\sqrt{5}}{5}$, so we will substitute in this fraction: 

$$\cos(2\theta) = 2\left(-\frac{\sqrt{5}}{5}\right)^2 - 1. \text{ This simplifies to: } 2\left(\frac{5}{25}\right) - 1 = \frac{2}{5} - 1 = -\frac{3}{5}.$$ 

Next I will find $\tan(2\theta)$ by using its formula: $\tan(2\theta) = \frac{2\tan \theta}{1 - \tan^2 \theta}$. We already know $\tan \theta = 2$, so we will substitute in this number: 

$$\tan(2\theta) = \frac{2(2)}{1 - (2)^2} = -\frac{4}{3}.$$ 

For the half angle formulas I need to determine which quadrant $\frac{\theta}{2}$ is in. To do this let’s first start with our given statement $180^\circ \leq \theta \leq 270^\circ$. If I divide everything by two we get: $90^\circ \leq \frac{\theta}{2} \leq 135^\circ$. This tells us that $\frac{\theta}{2}$ is in the second quadrant, so sine of $\frac{\theta}{2}$ should be positive, and the cosine and tangent of $\frac{\theta}{2}$ should be negative.

Now I will find $\sin \frac{\theta}{2}$ by using its formula: $\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}}$. We chose a positive because $\frac{\theta}{2}$ is in the second quadrant. We already know $\cos \theta = -0.8$, so we will substitute in this decimal:

$$\sin \frac{\theta}{2} = \sqrt{\frac{1 - \left(-\frac{\sqrt{5}}{5}\right)}{2}} = \sqrt{\frac{5 + \sqrt{5}}{10}} = \sqrt{\frac{5 + \sqrt{5}}{10}}.$$ 

Now I will find $\cos \frac{\theta}{2}$ by using its formula: $\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}}$. We chose a negative because $\frac{\theta}{2}$ is in the second quadrant. We already know $\cos \theta = -0.8$, so we will substitute in this decimal:

$$\cos \frac{\theta}{2} = \sqrt{\frac{1 + \left(-\frac{\sqrt{5}}{5}\right)}{2}} = \sqrt{\frac{5 - \sqrt{5}}{10}} = \sqrt{\frac{5 - \sqrt{5}}{10}}.$$
Finally I will find $\tan \frac{\theta}{2}$. I have three formulas to choose. I will choose: $\tan \frac{\theta}{2} = \frac{\sin \theta}{1+\cos \theta}$. We already know $\cos \theta = -0.8$, and $\sin \theta = 0.6$ so we will substitute in these decimals:

$$\tan \frac{\theta}{2} = \frac{-2\sqrt{5}}{5} = \frac{-2\sqrt{5}}{5 - \sqrt{5}} = \frac{-2\sqrt{5}}{5} \cdot \frac{5}{5 - \sqrt{5}} = \frac{-10\sqrt{5} - 10}{20} = \frac{-\sqrt{5} - 1}{2}.$$

EXAMPLE: Compute $\sin(22.5^\circ)$ and $\tan(22.5^\circ)$ using a half-angle formula.

We can write this as $\sin \left(\frac{45^\circ}{2}\right)$. Then we know that $\theta$ is 45 degrees, so now we use: $\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos 45^\circ}{2}}$. It is positive since 22.5 is in the first quadrant. You get: $\sin \frac{\theta}{2} = \sqrt{\frac{2 - \sqrt{2}}{2}} = \frac{\sqrt{2} - \sqrt{2}}{2} = \frac{\sqrt{2} - \sqrt{2}}{2}$.

For $\tan(22.5^\circ)$ we will use: $\tan \frac{\theta}{2} = \frac{-2\sqrt{5}}{5} = \frac{-2\sqrt{5}}{5 - \sqrt{5}} = \frac{-2\sqrt{5}}{5} \cdot \frac{5}{5 - \sqrt{5}} = \frac{-10\sqrt{5} - 10}{20} = \frac{-\sqrt{5} - 1}{2}$.

EXAMPLE: Rewrite $1 - 2\sin^2(\pi/12)$ using a double angle formula. Then find its exact value.

This one is written in the form $1 - 2\sin^2 \theta$, which is a form of $\cos 2\theta$. In this problem, we will let $\theta = \pi/12$.

Therefore we will substitute this into the formula $\cos 2\theta$: $\cos \left(2\left(\frac{\pi}{12}\right)\right)$. Simplifying gives us $\cos \frac{\pi}{6}$. Looking at our table we see this has an exact value of $\sqrt{3}/2$.

EXAMPLE: Establish the identity: $\frac{\cot \theta - \tan \theta}{\cot \theta + \tan \theta} = \cos 2\theta$

First we want to change these into sines and cosines.

$$\frac{\cos \theta - \sin \theta}{\sin \theta \cos \theta} = \cos 2\theta$$

Now get common denominators on the top and bottom.

$$\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta}$$

Multiply and write over a single denominator

$$\frac{\cos \theta}{\sin \theta} \cdot \frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} \cdot \frac{\sin \theta}{\sin \theta} = \cos 2\theta$$
\[
\frac{\cos^2 \theta - \sin^2 \theta}{\cos \theta \sin \theta} = \cos 2\theta
\]

We will get rid of the double fractions and use \(\cos^2 \theta + \sin^2 \theta = 1\)

\[
\cos^2 \theta - \sin^2 \theta = \cos 2\theta
\]

The left side is the identity for \(\cos 2\theta\).

\[
\cos 2\theta = \cos 2\theta
\]

EXAMPLE: \(4 \sin \theta \cos \theta)(1 - 2 \sin^2 \theta) = \sin 4\theta\)

First we will use the identity \(1 - 2 \sin^2 \theta = \cos 2\theta\). Now our problem becomes:

\[
(4 \sin \theta \cos \theta) \cos 2\theta = \sin 4\theta
\]

We know that \(\sin 2\theta = 2 \sin \theta \cos \theta\). We want to rewrite our problem as the following:

\[
2 \cdot 2 \sin \theta \cos \theta \cdot \cos 2\theta = \sin 4\theta
\]

Now we can use the identity \(\sin 2\theta = 2 \sin \theta \cos \theta\).

\[
2 \sin 2\theta \cos 2\theta = \sin 4\theta
\]

We know that \(2 \sin 2\theta \cos 2\theta\) is the same as \(\sin(2 \cdot 2\theta) = \sin 4\theta\).

\[
\sin 4\theta = \sin 4\theta
\]

EXAMPLE: Use power-reducing formulas to rewrite \(\cos^4 \theta\) in terms of first powers of cosine.

First we can rewrite the original problem as: \(\cos^2 \theta \cdot \cos^2 \theta\). This will allow us to use the power reducing formula for cosine. We will use the same formula twice:

\[
\frac{1 + \cos(2\theta)}{2} \cdot \frac{1 + \cos(2\theta)}{2} = \frac{1 + \cos^2(2\theta)}{4}.
\]

Now multiply across the top and bottom to get \(\frac{1 + 2 \cos(2\theta) + \cos^2(2\theta)}{4}\). We can split up the fraction: \(\frac{1}{4} + \frac{1}{2} \cos(2\theta) + \frac{1}{4} \cos^2(2\theta)\).

Now we will use another power reducing formula: \(\cos^2(2\theta) = \frac{1 + \cos(4\theta)}{2}\). When you substitute this in you will get \(\frac{1}{4} + \frac{1}{2} \cos(2\theta) + \frac{1}{4} \cdot \frac{1 + \cos(4\theta)}{2}\). Multiply to get \(\frac{1}{4} + \frac{1}{2} \cos(2\theta) + \frac{1 + \cos(4\theta)}{8}\). Now split up the last fraction: \(\frac{1}{4} + \frac{1}{2} \cos(2\theta) + \frac{1 + \cos(4\theta)}{8}\). Add the fractions to get the final answer: \(\frac{3}{8} + \frac{1}{2} \cos(2\theta) + \frac{1}{8} \cos(4\theta)\)