5.3 The Definite Integral

In the previous section, we used the same width for all of our rectangles when finding area. You don't always need to have the same width. Let $\|P\|$ be the norm, which represents the largest subinterval (partition), or the largest width. In the last section we added several rectangles together. If the widths are all the same, then $\|P\| = \Delta x = \frac{b-a}{n}$. We want the number of rectangles to go to infinity so we can get the exact area. Another way of looking at it is to have the norm go to zero. Now if the width of each rectangle goes to zero we can fit an infinite amount of them in our interval. So as $\|P\| \to 0$, $n \to \infty$.

The following expression can be used to find the exact area: $\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$. This is referred to as a Riemann Sum. This is also the definition for what is called the definite integral. This is basically an antiderivative but now we have an 'a' and a 'b'. So our definition notation will be: $\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx$.

EXAMPLE: Evaluate using the limit process: $\int_{-2}^{3} x \ dx$.

From this form we know that a = -2 and b = 3. Then we know $\|P\| = \Delta x = \frac{3-(-2)}{n} = \frac{5}{n}$. As $\|P\| \to 0$, $n \to \infty$. We also know that $f(x) = x$. We can also find our c_i, which is $c_i = -2 + \frac{5i}{n}$. We know that $f\left(-2 + \frac{5i}{n}\right) = -2 + \frac{5i}{n}$. Our limit formula is $\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$, which is also $\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$. From our information we know $\lim_{n\to\infty} \sum_{i=1}^{n} \left(-2 + \frac{5i}{n}\right)\frac{5}{n}$. We can multiply to get: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{-10}{n} + \frac{25i}{n^2}$. Now we can break up the summations: $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} -10 + \frac{25}{n^2}\sum_{i=1}^{n} i$. Using Theorem 4.2 we will get: $\lim_{n\to\infty} \frac{1}{n} \cdot -10n + \frac{25}{n^2} \cdot \frac{n^2+n}{2}$ Multiplying gives us: $\lim_{n\to\infty} -10 + \frac{25n^2 + 25n}{2n^2}$. Break up the fraction to get: $\lim_{n\to\infty} -10 + \frac{25}{2} + \frac{25}{2n}$. Taking the limit we get $-10 + \frac{25}{2} = \frac{5}{2}$.

EXAMPLE: Change the following into a definite integral: $\lim_{\|P\|\to 0} \sum_{i=1}^{n} 6c_i(4-c_i)^2 \Delta x_i$ where P is a partition of [0, 4].

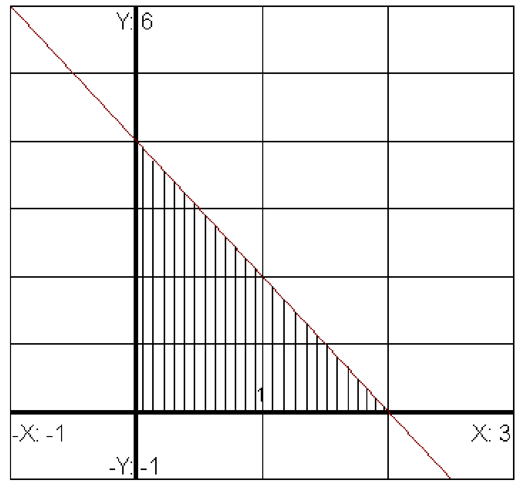
In our example we have a = 0 and b = 4. Our expression inside the summation is our f. If we put in an x for c_i then we have $f(x) = 6x(4-x)^2$. The width of the rectangle is really dx. So our answer is:

$$\int_0^4 6x(4-x)^2 \ dx$$

EXAMPLE: Change the following into a definite integral: $\lim_{\|P\|\to 0}\sum_{i=1}^{n}\left(\frac{3}{c_i^{\,2}}\right)\Delta x_i$ where P is a partition of [1, 3].

In our example we have a = 1 and b = 3. Our expression inside the summation is our f. If we put in an x for c_i then we have $f(x)=\frac{3}{x^2}$. The width of the rectangle is really dx. So our answer is: $\int_1^3 \frac{3}{x^2}\,dx$.

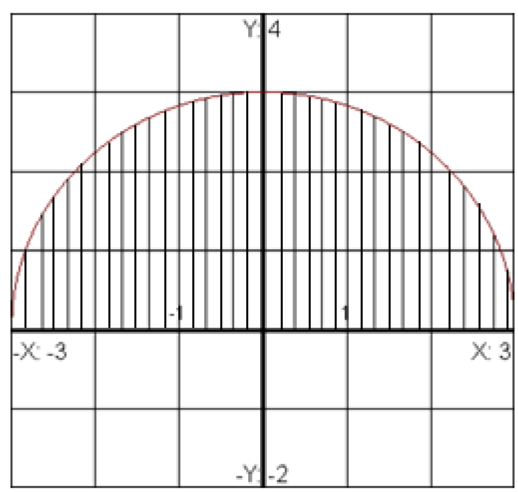
EXAMPLE: Graph the integrand $\int_0^2 -2x+4\ dx$, and use areas to evaluate the integral.



So we are really graphing the line $y=-2x+4$ since this is inside of the integrand. We plot (0, 4) and from there we go down two units and go one to the right since –2 is the slope.   You will get the graph shown: The shaded region would be between 0 and 2. We shade everything below the line and above the x-axis between 0 and 2.

The shaded region results in a triangle, so we can use the area formula $A=\frac{1}{2}bh$. The base is 2 and the height is 4. So $A=\frac{1}{2}(2)(4)$.
The area is 4. This would be the exact area.

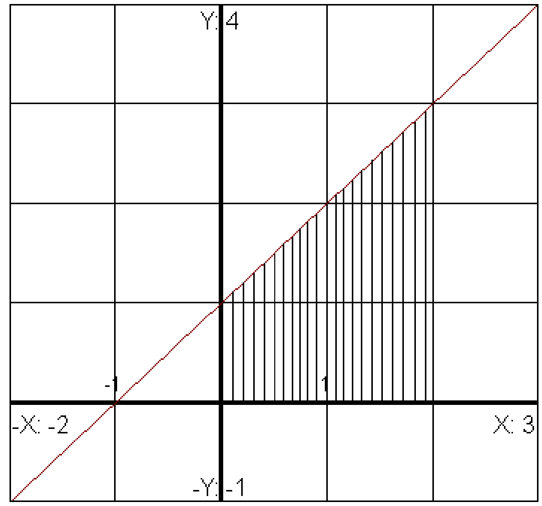
EXAMPLE: Graph the integrand $\int_{-3}^{3}\sqrt{9-x^2}\ dx$, and use areas to evaluate the integral.



The graph of this is a semi-circle, but if you did not know how to graph it then you can use a table of values. This graph is only defined between –3 and 3, so your table of x values can only be between –3 and 3.
The graph is shown, and notice that the shaded region is below the curve and above the x-axis between the values of –3 and 3.

So since we have a half-circle, the area formula is $A=\frac{1}{2}\pi r^2$. I just took the area of a circle and multiplied it by a half since we have a semicircle.
So the area is: $A=\frac{1}{2}\pi(3)^2$. This simplifies to $A=\frac{9\pi}{2}$ which would be the exact area.

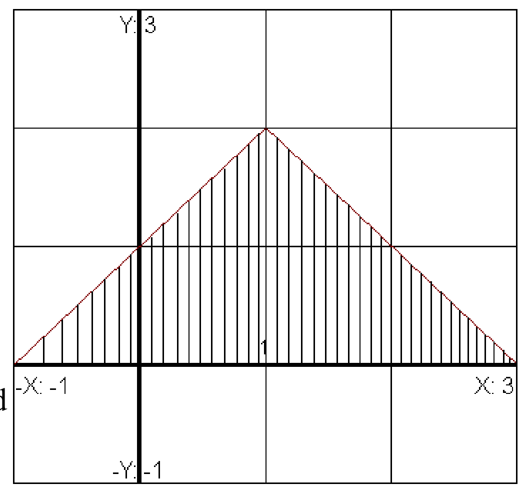
EXAMPLE: Graph the integrand $\int_0^2 x+1 \ dx$, and use areas to evaluate the integral.



So we are really graphing the line $y = x + 1$ since this is inside of the integrand. We plot (0, 1) and from there we go up one unit and one unit to the right since 1 is the slope. You will get the graph shown: The shaded region would be between 0 and 2. We shade everything below the line and above the x-axis between 0 and 2.

We need to do two different areas, one for the triangle and one for a rectangle. Then we will add these together. For the triangle, we can use the formula $A = \frac{1}{2}bh$. The base is 2 and the height is 2. So $A = \frac{1}{2}(2)(2)$. The area is 2. We also have a rectangle with a length of 2 and height of 1, so A = (2)(1) = 2. The total area of the shaded region will be 2 + 2, which is 4.

EXAMPLE: Graph the integrand $\int_{-1}^{3} 2-|x-1| \ dx$, and use areas to evaluate the integral.



This is an absolute value because of the V shape. We can rewrite the inside as: $y = -|x-1| + 2$ The graph is drawn by starting with the base graph of $y = |x|$ and using transformations. The base graph is shifted to the right one unit and up 2 units. Then it is flipped over the horizontal axis because of the minus sign in front of the absolute value. The shaded portion of the graph is below the line and above the x-axis, between –1 and 3. For the area, we have a triangle with a base of a 4 and a height of 2. The area is $A = \frac{1}{2}(4)(2) = 4$.

Properties of Definite Integrals (Assume f and g are integrable on [a, b]).

1.) $\int_a^a f(x)dx = 0$ From a to a we have a rectangle with a width of 0, so the area is 0.

2.) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ If we switch the order of a and b the area changes sign.

3.) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ We can split up the area in to two separate areas.

4.) $\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$ We are allowed to take a constant k out of the integral.

5.) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$ We can do two separate integrals.

EXAMPLE: Given $\int_2^4 x^3 \ dx = 60$, $\int_2^4 x \ dx = 6$, and $\int_2^4 dx = 2$ evaluate the following using properties of definite integrals:

a.) $\int_2^2 x^3 \ dx$

This is using property number one, so the answer is 0.

b.) $\int_2^4 15 \ dx$

First we will use property #4 to take out the constant 15: $15\int_2^4 dx$. From our given information, $\int_2^4 dx = 2$, so our answer is 15(2) = 30.

c.) $\int_2^4 \left(x^3 + 4\right) dx$

We can use property #5 to break this up and property #4 to take out the constant 4. You will get $\int_2^4 x^3 dx + 4\int_2^4 dx$. Now put in our given information: 60 + 4(2) = 68.

d.) $\int_4^2 x(2 - x^2) \ dx$

First we want to switch the limits of integration (2 and 4) so that it matches what we are given (property #2). I will also multiply the expression. You will get: $-\int_2^4 2x - x^2 \ dx$. Now we will break up the integral and distribute the negative: $-2\int_2^4 x \ dx + \int_2^4 x^3 \ dx$. Now we put in our given information: -2(6) + 60 = 48.

EXAMPLE: Given $\int_1^3 f(x)\ dx = -2$, $\int_1^6 f(x)\ dx = 5$, and $\int_1^6 g(x)\ dx = 7$, evaluate the following using properties of definite integrals:

a.) $\int_3^3 g(x)\ dx$

This is using property number one, so the answer is 0.

b.) $\int_6^1 3f(x)\ dx$

We can use property #4 to take out the 3: $3\int_6^1 f(x)\ dx$. Now we will switch the limits and use property #2:

$-3\int_1^6 f(x)\ dx$. We can now put in our answer from part a: -3(5) = –15.

c.) $\int_3^6 f(x)\ dx$

We need to rewrite this in terms of values given. We can rewrite this as: $\int_3^6 f(x)\ dx = \int_1^6 f(x)\ dx - \int_1^3 f(x)\ dx$.

So the answer is 5 – (–2) = 7.

d.) $\int_6^1 [f(x) - g(x)]\ dx$

The integrands can be broken up: $\int_6^1 [f(x) - g(x)]\ dx = \int_6^1 f(x)\ dx - \int_6^1 g(x)\ dx$. However we need to flip the numbers on the integrand, so we apply another property: $-\int_1^6 f(x)\ dx + \int_1^6 g(x)\ dx$. So we have values for each of these integrands, so our answer is: –5 + 7 = 2.

e.) $\int_1^6 [-2f(x) + g(x)]\ dx$

The integrands can be broken up: $\int_1^6 [-2f(x) + g(x)]\ dx = \int_1^6 -2f(x)\ dx + \int_1^6 g(x)\ dx$. For the first integrand we will take out the –3: $-2\int_1^6 f(x)\ dx + \int_1^6 g(x)\ dx$. Now put in our given information: –2(5) + 7 = –3.