5.4 The Fundamental Theorem of Calculus

First Fundamental Theorem of Calculus

If a function f is continuous on the closed interval [a, b] and F is antiderivative of f on the interval [a, b] then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

EXAMPLE: Evaluate: $\int_1^3 -3x^2 + 12x - 9\ dx$.

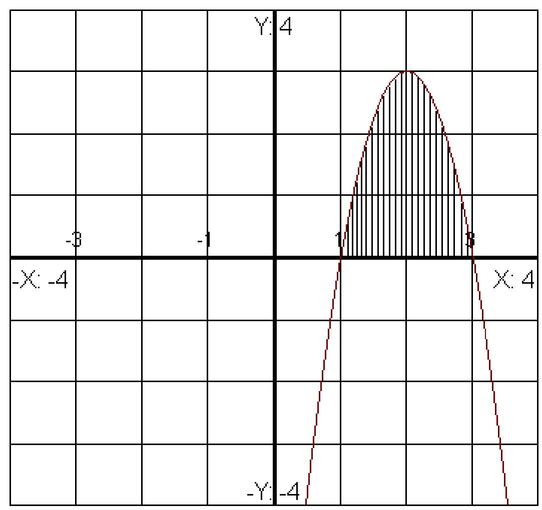
First we need to take the antiderivative: $-3\cdot\frac{x^3}{3} + 12\cdot\frac{x^2}{2} - 9x$. There is no C in these problems because we are going to find the numerical answer. Now there is some new notation we are going to use: $-x^3 + 6x^2 - 9x\Big|_1^3$
The solid line means that we are ready to plug in 1 and 3 into the antiderivative and then subtract F(3) – F(1):

$$-x^3 + 6x^2 - 9x\Big|_1^3 = \left[-(3)^3 + 6(3)^2 - 9(3)\right] - \left[-(1)^3 + 6(1)^2 - 9(1)\right]$$

$$-x^3 + 6x^2 - 9x\Big|_1^3 = [0] - [-4] = 4.$$

Therefore, $\int_1^3 -3x^2 + 12x - 9\ dx = 4$.

The graph to the right shows the actual area we are calculating.



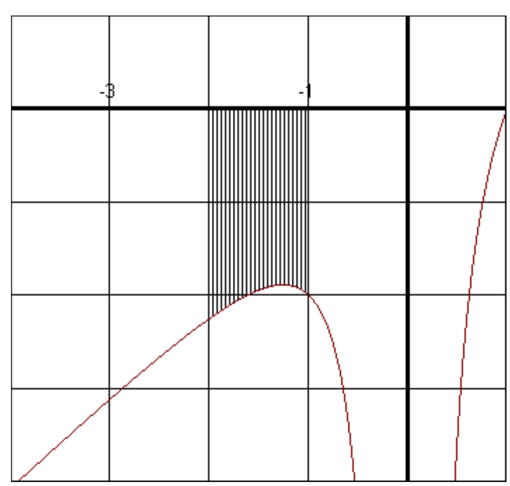
EXAMPLE: Evaluate: $\int_{-2}^{-1} u - \frac{1}{u^2}\ du$.

We can rewrite this as $\int_{-2}^{-1} u - u^{-2}\ du$. After taking the antiderivative we will get $\frac{u^2}{2} - \frac{u^{-1}}{-1}\Big|_{-2}^{-1}$, which simplifies to: $\frac{u^2}{2} + \frac{1}{u}\Big|_{-2}^{-1}$.

$$\frac{u^2}{2} + \frac{1}{u}\Big|_{-2}^{-1} = \left[\frac{(-1)^2}{2} + \frac{1}{-1}\right] - \left[\frac{(-2)^2}{2} + \frac{1}{-2}\right]$$

$$\frac{u^2}{2} + \frac{1}{u}\Big|_{-2}^{-1} = \left[-\frac{1}{2}\right] - \left[\frac{3}{2}\right] = -2.$$

The reason why this is negative is because the area is below the x-axis, as shown in the graph:



EXAMPLE: Evaluate: $\int_1^8 \frac{3x^2 - \sqrt[3]{x^2}}{3x^2}\ dx$.

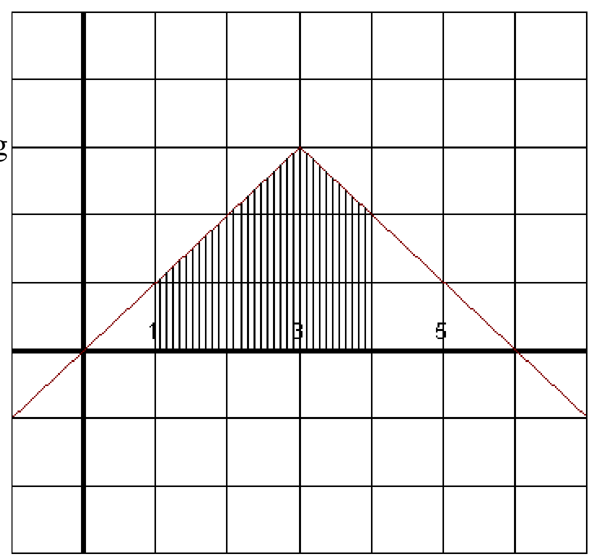
We can rewrite this as: $\int_1^8 \frac{3x^2 - x^{\frac{2}{3}}}{3x^2}\ dx$. Now divide everything in the numerator by the denominator:

$\int_1^8 \frac{3x^2}{3x^2} - \frac{x^{\frac{2}{3}}}{3x^2}\ dx$. This simplifies to: $\int_1^8 1 - \frac{1}{3}x^{-\frac{4}{3}}\ dx$. Now we can take the antiderivative: $x - \frac{1}{3} \cdot \frac{x^{-\frac{1}{3}}}{-\frac{1}{3}}\Big|_1^8$. This simplifies to: $x + \frac{1}{x^{\frac{1}{3}}}\Big|_1^8$. Now we plug in the 8 and 1:

$x + \frac{1}{\sqrt[3]{x}}\Big|_1^8 = \left[8 + \frac{1}{\sqrt[3]{8}}\right] - \left[1 + \frac{1}{\sqrt[3]{1}}\right]$. This becomes:

$\left[8 + \frac{1}{2}\right] - [1 + 1] = \frac{13}{2}$.

EXAMPLE: Evaluate: $\int_1^4 3 - |x - 3|\ dx$.



Let's look at the graph of this to the right. The absolute value can be broken up into two lines. The positive sloping line has an equation of $y = x$. The negative sloping line has an equation of $y = -x + 6$. We are going to split up the area into two separate ones since we have two different equations. $\int_1^3 x\ dx + \int_3^4 -x + 6\ dx$. Now we will evaluate each one separately:

$\frac{x^2}{2}\Big|_1^3 + -\frac{x^2}{2} + 6x\Big|_3^4$

$\left[\frac{(3)^2}{2}\right] - \left[\frac{(1)^2}{2}\right] + \left(\left[-\frac{4^2}{2} + 6(4)\right] - \left[-\frac{(3)^2}{2} + 6(3)\right]\right)$

$\frac{9}{2} - \frac{1}{2} + \left(16 - \frac{27}{2}\right) = 4 + \frac{5}{2} = \frac{13}{2}$.

EXAMPLE: Evaluate: $\int_0^{\frac{\pi}{4}} \frac{1-\sin^2\theta}{\cos^2\theta}\,d\theta$.

We will use the identity $\cos^2\theta = 1-\sin^2\theta$ to simplify this to: $\int_0^{\frac{\pi}{4}} \frac{\cos^2\theta}{\cos^2\theta}\,d\theta$. This equals $\int_0^{\frac{\pi}{4}} 1\ d\theta$. When we do the antiderivative we will get: $\theta\Big|_0^{\frac{\pi}{4}} = \left[\frac{\pi}{4}\right]-[0] = \frac{\pi}{4}$.

EXAMPLE: Evaluate: $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2-\csc^2\theta\ d\theta$.

The antiderivative is: $2\theta + \cot\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$. We put in $\frac{\pi}{2}$ and then $\frac{\pi}{4}$. You will get:

$$\left[2\cdot\frac{\pi}{2}+\cot\frac{\pi}{2}\right]-\left[2\cdot\frac{\pi}{4}+\cot\frac{\pi}{4}\right] = [\pi+0]-\left[\frac{\pi}{2}+1\right] = \frac{\pi}{2}-1.$$

EXAMPLE: Evaluate: $\int_1^2 \frac{1}{x}-2e^{2x}\ dx$.

Recall the antiderivative of $\frac{1}{x}$ is $\ln|x|$. To take the antidervative of $2e^{2x}$ we need to use the formula $\frac{1}{k}e^{kx}$ where k = 2 in this case. So the antiderivative is: $\ln|x| - 2\cdot\frac{1}{2}e^{2x}\Big|_1^2$. This simplifies to $\ln|x| - e^{2x}\Big|_1^2$. We put in 2 and then 1. You will get: $\left[\ln|2| - e^{2\cdot 2}\right]-\left[\ln|1| - e^{2\cdot 1}\right] = \left[\ln 2 - e^4\right]-\left[0-e^2\right] = \ln 2 - e^4 + e^2$.

EXAMPLE: Evaluate: $\int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}} \frac{1}{x\sqrt{4x^2-1}}\ dx$.

First we will rewrite this as: $\int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}} \frac{1}{x\sqrt{2^2x^2-1}}\ dx$. So we will use this formula: $\int \frac{1}{x\sqrt{k^2x^2-1}}\ dx = \sec^{-1}kx$ with $k = 2$. So the antiderivative is: $\sec^{-1}2x\Big|_{1/2}^{\frac{1}{\sqrt{3}}}$. So now we plug in our limits of integration:

$$\sec^{-1}2x\Big|_{1/2}^{\frac{1}{\sqrt{3}}} = \left[\sec^{-1}\left(2\cdot\frac{1}{\sqrt{3}}\right)\right]-\left[\sec^{-1}\left(2\cdot\frac{1}{2}\right)\right] = \left[\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)\right]-\left[\sec^{-1}(1)\right] = \frac{\pi}{6}-0 = \frac{\pi}{6}$$

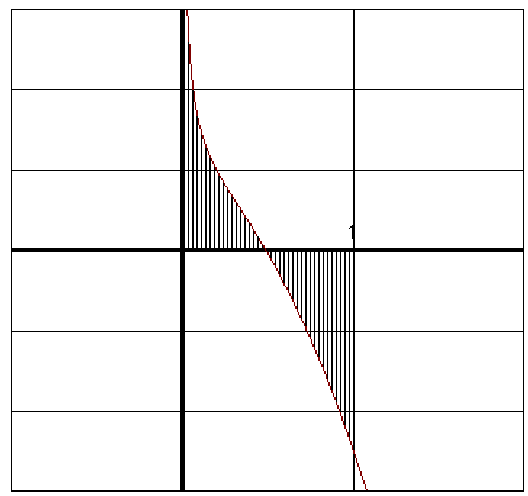
EXAMPLE: Evaluate: $\int_{-1}^{2} 2^x \, dx$.

In order to find the antiderivative we need to use the formula: $\left(\frac{1}{k \ln a}\right) a^{kx}$ where $k = 1$ and $a = 2$ in this case.

Following the formula we get: $\left(\frac{1}{\ln 2}\right) 2^x$. So now we are ready to evaluate:

$$\left.\frac{2^x}{\ln 2}\right|_{-1}^{2} = \left[\frac{2^2}{\ln 2}\right] - \left[\frac{2^{-1}}{\ln 2}\right] = \frac{4}{\ln 2} - \frac{1}{2\ln 2} = \frac{7}{2\ln 2}. \qquad \text{So } \int_{-1}^{2} 2^x \, dx = \frac{7}{2\ln 2}.$$

EXAMPLE: Evaluate: $\int_0^1 \frac{1}{2\sqrt{x}} - 3x^2 \, dx$

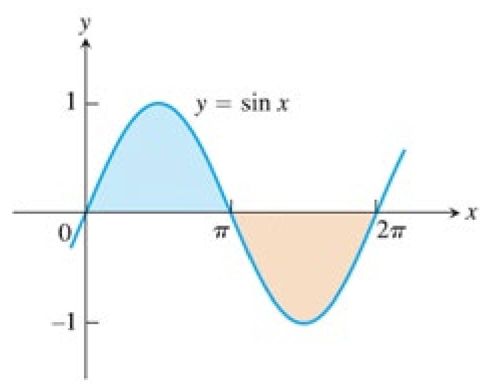


We can rewrite this as $\int_0^1 \frac{1}{2} x^{-\frac{1}{2}} - 3x^2 \, dx$. Now we take the antiderivative: $\left.\frac{\frac{1}{2} x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{3x^3}{3}\right|_0^1$. This simplifies to:

$$\left. x^{\frac{1}{2}} - x^3 \right|_0^1 = \left[(1)^{\frac{1}{2}} - (1)^3\right] - \left[(0)^{\frac{1}{2}} - (0)^3\right] = [0] - [0] = 0$$

As you can see in the graph to the right, we have the same amount of area above the graph as below, so they cancel out. This is why the area is zero.

EXAMPLE: Evaluate: $\int_0^{2\pi} \sin x \, dx$

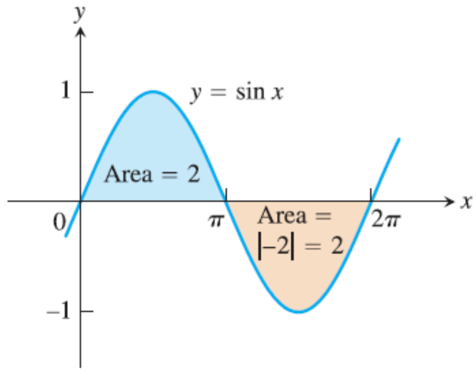


First we will take the antiderivative: $-\cos x\Big|_0^{2\pi}$. Now plug in the limits of the integrand: $-\cos(2\pi) - (-\cos 0)$. Since $\cos(2\pi) = \cos(0) = 1$ we can plug these into our expression: $-1 - (-1) = 0$. We can see in the picture that we have equal areas on the top and bottom which cancel each other out, which is why we get 0 as our answer. However we need to be careful of how the question is asked as shown in the next example. If it asks us to find the total area between the graph and the x-axis over an interval, then this requires a certain procedure.

To find the total area between the graph of $y = f(x)$ and the x-axis over the interval [a, b]:

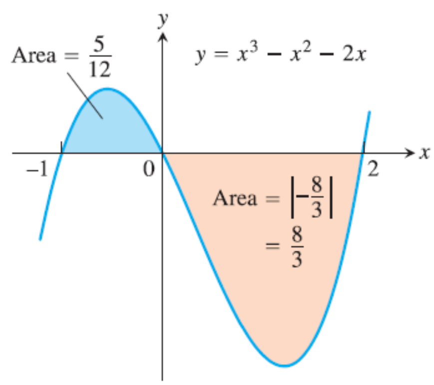
1.) Subdivide [a, b] at the zeros of *f*.
2.) Integrate f over each subinterval.
3.) Add the absolute values of the integrals.

EXAMPLE: Find the total area between the graph of $y = \sin x$ and the x-axis over the interval $[0, 2\pi]$.



This may seem like the exactly same problem as the previous definite integral, however it is not. Whenever the question is worded this way without the integral symbol, you must follow the above three steps. We know the graph crosses the x-axis at $0,\ \pi,\ 2\pi$ as shown in the graph here. This means we need to split up our interval into the following: $\int_0^{2\pi} \sin x\ dx = \left|\int_0^{\pi} \sin x\ dx\right| + \left|\int_{\pi}^{2\pi} \sin x\ dx\right|$. We will find the antiderivative: $\left|-\cos x\Big|_0^{\pi}\right| + \left|-\cos x\Big|_{\pi}^{2\pi}\right|$. Then we will evaluate each one: $|-\cos(0) - (-\cos\pi)| + |-\cos(\pi) - (-\cos 2\pi)|$. We can simplify: $|-\cos(0) + \cos\pi| + |-\cos(\pi) + \cos 2\pi|$. Then after substituting values: $|-1 + -1| + |-(-1) + 1| = |-2| + |2| = 2 + 2 = 4$. So even though we are looking at the same function, the answers different because of the way the question was originally asked.

EXAMPLE: Find the total area between the graph of $y = x^3 - x^2 - 2x$ and the x-axis if $-1 \le x \le 2$.



Because this is asking for the total area, we will need to break this up according to the x-intercepts as shown in the diagram. To find the x-intercepts we first need to factor by taking out an x: $y = x(x^2 - x - 2)$. Next we factor again to get $y = x(x+1)(x-2)$. When you set this equal to zero you will get x = –1, 0 and 2. We now want to break up the integral according to the x-intercepts. You will get:

$$\int_{-1}^{2} x^3 - x^2 - 2x\ dx = \left|\int_{-1}^{0} x^3 - x^2 - 2x\ dx\right| + \left|\int_{0}^{2} x^3 - x^2 - 2x\ dx\right|.$$

We will find the antiderivative: $\left|\frac{x^4}{4} - \frac{x^3}{3} - x^2\Big|_{-1}^{0}\right| + \left|\frac{x^4}{4} - \frac{x^3}{3} - x^2\Big|_{0}^{2}\right|$. Then we will evaluate each one:

$$\left|\left(\frac{0^4}{4} - \frac{0^3}{3} - 0^2\right) - \left(\frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2\right)\right| + \left|\left(\frac{2^4}{4} - \frac{2^3}{3} - 2^2\right) - \left(\frac{(0)^4}{4} - \frac{(0)^3}{3} - (0)^2\right)\right|.$$

We can simplify:

$$\left|(0) - \left(-\frac{5}{12}\right)\right| + \left|-\frac{8}{3} - (0)\right| = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}.$$

EXAMPLE: Find: $\frac{d}{dx}\int_4^x \sin\theta \; d\theta$.

We need to find the integral and then take the derivative of our answer. First let's do the integral: $\frac{d}{dx}\left[-\cos\theta\Big|_4^x\right] = \frac{d}{dx}[-\cos x - (-\cos 4)]$. So now we want to do $\frac{d}{dx}[-\cos x + \cos 4]$. This means we are taking the derivative. You will get: $\sin x$. The derivative of $\cos 4$ is zero since it is a constant. Therefore, $\frac{d}{dx}\int_4^x \sin\theta \; d\theta = \sin x$. Notice that the derivative is the same as what we had in the original integrand. Also notice that the 4 in our problem could have been any constant, and that wouldn't effect our answer. The reason why this works is because of the Second Fundamental Theorem of Calculus.

Second Fundamental Theorem of Calculus

Assume f is continuous on an open interval I containing a, then, for every x in the interval, the following is true: $\frac{d}{dx}\left[\int_a^{g(x)} f(t) \; dt\right] = f(g(x)) \cdot g'(x)$. Notice the value of a does not affect our answer.

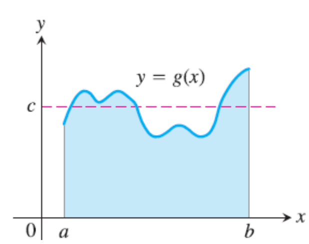
EXAMPLE: Find $f'(x)$ if $f(x) = \int_{-1}^{x} \frac{t^2}{t^2+1} dt$.

Using the Second Fundamental Theorem of Calculus, just replace the t with an x. Then we will multiply this by the derivative of x, which is one. You will get: $f'(x) = \frac{x^2}{x^2+1}(1)$ or $f'(x) = \frac{x^2}{x^2+1}$.

EXAMPLE: Find $f'(x)$ if $f(x) = \int_0^{x^2} \sin\theta^2 \; d\theta$.

For this one we will follow the formula. We will replace the theta with x squared. Then we will multiply this by the derivative of x squared, which is 2x. Our answer will look like: $f'(x) = \sin(x^2)^2 \cdot 2x$. So our final answer is: $f'(x) = 2x\sin\left(x^4\right)$.

Average Value of a Continuous Function



If you want to take an average of a list of items you would add them all up and then divide by number of items. Suppose you wanted to find the average value of a continuous function *f* on an interval [*a*, *b*]. The graph to the right illustrates this idea. The graph of *g*(*x*) has an average height of *c* between *a* and *b*. So geometrically the average (mean) value of *g*(*x*) on [a, b] is the area under the graph divided by *b* – *a*. The area under a graph can be represented with a definite integral. So we will take the definite integral and divide by *b* – *a*.

Here is the notation: av(*f*) = $\frac{1}{b-a}\int_a^b f(x) \; dx$.

EXAMPLE: Graph $f(x) = -3x^2 - 1$ and find its average value over the interval $[0, 1]$.

We need to set up the correct integral using the formula. Here, a = 0 and b = 1:

av(f) = $\dfrac{1}{1-0}\int_0^1 -3x^2 - 1\,dx$. To do this we will take the antiderivative:

av(f) = $\dfrac{1}{1}\left(-3 \cdot \dfrac{x^3}{3} - x \Big|_0^1 \right)$ Now simplify.

av(f) = $-x^3 - x\Big|_0^1$ Put in the limits of integration.

av(f) = $\left(-(1)^3 - (1)\right) - \left(-(0)^3 - (0)\right)$ Now simplify.

av(f) = $-1 - 1 = -2$

So the average value is negative two.

Now let's look at the graph: