5.6 Substitution with Definite Integrals

In this section we will focus on definite integrals using substitution. First find the indefinite integral using substitution and then plug in the limits of integration. We will not cover the area between curves.

EXAMPLE: Integrate by substitution: $\int_0^4 x \cdot \sqrt{9+x^2}\ dx$.

Since this one has the 0 and 4 this means we will get a numerical answer. We do everything the same as before and then plug in the limits. First we will let $u = 4 + x^2$. Then $du = 2x\ dx$. Solving for dx we get $dx = \frac{du}{2x}$.

Now we will substitute this for dx and we will substitute a u for $4 + x^2$. Now we have $\int x \cdot u^{\frac{1}{2}} \cdot \frac{du}{2x}$. We get

$\frac{1}{2}\int u^{\frac{1}{2}} du$. The antiderivative is $\frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}}$. This simplifies to: $\frac{1}{3}u^{\frac{3}{2}}$ Now substitute back in the x and get ready to

plug in the 0 and 4: $\frac{1}{3}(9+x^2)^{\frac{3}{2}}\Big|_0^4$. It is VERY IMPORTANT to know that if you are going to plug in the 0 and 4 you must substitute the x for u since 0 and 4 are x values. If you want to use the u, then you must change the 0 and 4 into u values by using $u = 4 + x^2$. In our problem we will just use 0 and 4:

$$\frac{1}{3}(9+x^2)^{\frac{3}{2}}\Big|_0^4 = \left[\frac{1}{3}(9+4^2)^{\frac{3}{2}}\right] - \left[\frac{1}{3}(9+0^2)^{\frac{3}{2}}\right] = \left[\frac{1}{3}(25)^{\frac{3}{2}}\right] - \left[\frac{1}{3}(9)^{\frac{3}{2}}\right] = \left[\frac{1}{3}(125)\right] - \left[\frac{1}{3}(27)\right] = \frac{125}{3} - 9 = \frac{98}{3}.$$

EXAMPLE: Integrate by substitution: $\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{(3+2\cos\theta)^2}\ d\theta$.

First we will let $u = 3 + 2\cos\theta$. Then $du = -2\sin\theta\ d\theta$. Solving for $d\theta$ we get $d\theta = \frac{du}{-2\sin\theta}$. Now we will substitute this for $d\theta$ and we will substitute a u for $3 + 2\cos\theta$. Now we have $\int \frac{\sin\theta}{(u)^2} \cdot \frac{du}{-2\sin\theta}$. Simplifying give us $-\frac{1}{2}\int \frac{du}{(u)^2} = -\frac{1}{2}\int u^{-2} du$. Integrating gives us $-\frac{1}{2} \cdot \frac{u^{-1}}{-1} = \frac{1}{2u}$. Now substitute back in the x and get ready to plug in the limits of integration: $\frac{1}{2(3+2\cos\theta)}\Big|_0^{\frac{\pi}{2}} = \left[\frac{1}{2\left(3+2\cos\frac{\pi}{2}\right)}\right] - \left[\frac{1}{2(3+2\cos(0))}\right] =$

$\left[\frac{1}{2(3+2(0))}\right] - \left[\frac{1}{2(3+2(1))}\right]$. Now we will simplify: $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$.

EXAMPLE: Integrate by substitution: $\int_{\frac{\pi}{4}}^{\pi} \sqrt{\tan\theta}\sec^2\theta\ d\theta$.

First we will let $u = \tan\theta$. Then $du = \sec^2\theta\ d\theta$. Solving for $d\theta$ we get $d\theta = \dfrac{du}{\sec^2\theta}$. Now we will substitute this for $d\theta$ and we will substitute a *u* for $\tan\theta$. Now we have $\int \sqrt{u}\sec^2\theta \cdot \dfrac{du}{\sec^2\theta}$. We get $\int u^{\frac{1}{2}}\ du$.

The antiderivative is $\dfrac{u^{\frac{3}{2}}}{\frac{3}{2}}$. This simplifies to: $\dfrac{2}{3}u^{\frac{3}{2}}$. This time I will do this one differently by not substituting the x back in. If you do not substitute then we need to convert our limits of integration because the original ones are for x. Earlier we said that $u = \tan\theta$. So we will plug in π and $\dfrac{\pi}{4}$: $u = \tan\pi = 0$ and $u = \tan\dfrac{\pi}{4} = 1$.

So now that we have our u values we have: $\left.\dfrac{2}{3}u^{\frac{3}{2}}\right|_1^0$. Now we will plug in the limits of integration:

$\left[\dfrac{2}{3}(0)^{\frac{3}{2}}\right] - \left[\dfrac{2}{3}(1)^{\frac{3}{2}}\right] = 0 - \dfrac{2}{3} = -\dfrac{2}{3}$.

EXAMPLE: Find the definite integral: $\int_0^1 \dfrac{1}{\sqrt{4-x^2}}\ dx$.

We can rewrite this problem as: $\int_0^1 \dfrac{1}{\sqrt{2^2-x^2}}\ dx$ Integrating we will use this formula with a = 2:

$\int \dfrac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\dfrac{u}{a} + C$. We will get: $\left.\sin^{-1}\dfrac{x}{2}\right|_0^1$. Now we put in the limits: $\left[\sin^{-1}\dfrac{1}{2}\right] - \left[\dfrac{1}{2}\sin^{-1}\dfrac{0}{2}\right]$. We get: $\dfrac{\pi}{6} - 0 = \dfrac{\pi}{6}$.

EXAMPLE: Find the indefinite integral: $\int_e^{e^2} \dfrac{1}{x\ln x}\ dx$.

Using substitution we will get $u = \ln x$. Then $du = \dfrac{1}{x}\ dx$. Solving for dx we get: $dx = x\ du$. Now we make our substitution: $\int_e^{e^2} \dfrac{1}{x \cdot u} \cdot x\ du$. This simplifies to: $\int_e^{e^2} \dfrac{1}{u}\ du$. When we integrate this we get $\ln|u|$. Then we can replace the u with $\ln x$ and we get: $\left.\ln|\ln x|\right|_e^{e^2}$. Now we plug in our top limit and our bottom limit: $\ln|\ln e^2| - \ln|\ln e|$. This can be rewritten as: $\ln|2\ln e| - \ln|\ln e|$. We also know that $\ln e = 1$, so our problem becomes: $\ln|2| - \ln|1|$. Since $\ln 1 = 0$ our answer is $\ln 2$.