2.1 Functions

Domain: (input) all the x-values that make the equation defined
Defined: There is no division by zero or square roots of negative numbers
Range: (output) all y-values that a graph uses.

Function Definition: For each input (x) there can only be one output (y).

EXAMPLE: For each relation below, determine whether it is a function. Then give the domain and range for each relation.

{(1, 2), (3, 7), (2, 9), (8, 11)}

This is a function. Every x goes to only one y.
The domain of this is all the x-values. The answer is {1, 3, 2, 8}. You may leave it this way or order them.
The range of this is all the y-values. The answer is {2, 7, 9, 11}.

{(-3, 4), (5, 6), (7, 4), (-2, 3)}

This is a function. Even though the x-values -3 and 7 both go to 4, each x value goes to only one y-value.
Domain: {-3, 5, 7, -2}
Range: {4, 6, 3} Notice that 4 is repeated, but you only need to write it once.

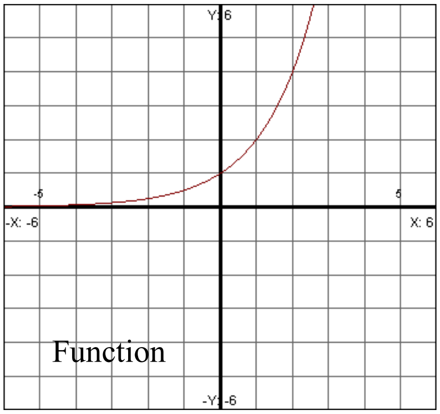
{(-2, 4), (-1, 6), (0, 3), (-2, 8)}

NOT a function because when x is -2 it goes to both 4 and 8. There are two different y values for one x.
Domain: {-2, -1, 0} Notice again that even though -2 is repeated, it only needs to be written once.
Range: {4, 6, 3, 8}

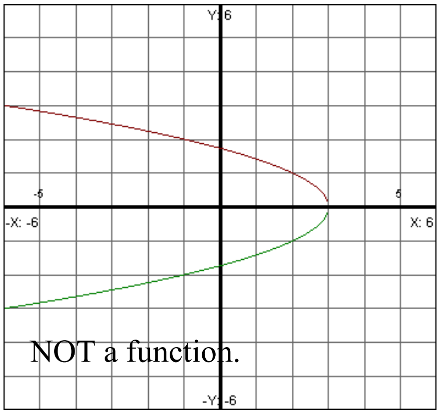
{(5, 3), (-2, 1), (5, 3), (9, 10)}

This is a function. The same point repeats, but still goes to only one y.
Domain: {5, -2, 9}
Range: {3, 1, 10}

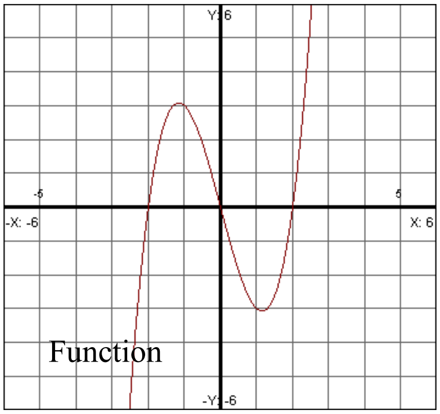
<u>**Vertical line test**</u>. If you pass an imaginary vertical line through the graph and it only intersects the graph once then it is a function. Which graphs below are functions?



Function



NOT a function.



Function

EXAMPLE: Is $x^2 + 3y^2 = 7$ a function?

We don't have a graph drawn for us or a set of points. We need to see if it is a function algebraically. First think we need to do is solve for y. We will isolate it and then take the square root of both sides. Don't forget that you will get a plus and minus whenever you take the even root of something.

$$3y^2 = 7 - x^2$$
$$y^2 = \frac{7 - x^2}{3}$$

$y = \pm\sqrt{\frac{7 - x^2}{3}}$ Notice that for each x we will get two different y values because of the $\pm$. Therefore we know this is not a function.

EXAMPLE: Is $x^2 + y^5 = 1$ a function?

Again we will solve for y. When we do we will take the odd root of both sides. There will be no plus and minus here since it was an odd root.

$$y^5 = 1 - x^2$$

$y = \sqrt[5]{1 - x^2}$ To get rid of the fifth power, I took the fifth root. For each x we put in we will only get one y-value for each x we put in so it IS a function.

So what is the general rule here based on our previous two examples?

Any equation that has a y raised to an even power is NOT a function.
Any equation that has a y raised to an odd power IS a function.

Function notation: $f(x)$ which means "f of x". This does not mean f times x. It means that we have a function called f which contains the variable x.

EXAMPLE: Given the function $f(x) = 2x - 5$, find the following:

a.) Find $f(3)$. Solve $f(x) = 7$.

Whatever is inside the parenthesis goes in place of x in the original expression. This is really asking us for the y value when x is 3.

$$f(3) = 2(3) - 5$$
$$f(3) = 1$$

To solve $f(x) = 7$, we put in a 7 for $f(x)$: $7 = 2x - 5$. Now we solve for x. Add the 5 to both sides and divide both sides by 2 to get the answer of 6.

b.) $f(x+3)$

Now we need to replace x in the original equation with x + 3. Then simplify.

$f(x+3) = 2(x+3) - 5$
$f(x+3) = 2x + 6 - 5$
$f(x+3) = 2x + 1$ This is as far as we can go on this one.

c.) $f(x) + f(3)$

For this one we can replace the $f(x)$ with 2x – 5. We also know $f(3)$.

$f(x) + f(3) = 2x - 5 + 1$
$f(x) + f(3) = 2x - 4$ Notice this is not the same as part b, so the f is not distributed to the x and 3.

d.) $f(x+h)$

For this one just replace the x with the expression x + h.

$f(x+h) = 2(x+h) - 5$
$f(x+h) = 2x + 2h - 5$ This is as far as we can go.

EXAMPLE: Let $f(x) = \dfrac{x-4}{2x-3}$. Find the following:

a.) $f(5)$.

$f(5) = \dfrac{5-4}{2(5)-3}$ We are replacing x with 5.

$f(5) = \dfrac{1}{7}$

b.) $f(x+h)$

$f(x+h) = \dfrac{(x+h)-4}{2(x+h)-3}$ We are replacing x with the quantity (x + h).

$f(x+h) = \dfrac{x+h-4}{2x+2h-3}$ This is as far as we can go.

c.) $f(x)+f(5)$

$\frac{x-4}{2x-3}+\frac{1}{7}$ We are replacing f(x) with our original function and f(5) we found in part a.

$\left(\frac{7}{7}\right)\frac{x-4}{2x-3}+\left(\frac{2x-3}{2x-3}\right)\frac{1}{7}$ Generally if you have two fractions, then combine after common denominators.

$\frac{7(x-4)+(2x-3)}{7(2x-3)}$ Now add the fractions together now that we have common denominators.

$\frac{7x-28+2x-3}{7(2x-3)}$ Distribute and simplify.

$\frac{9x-31}{7(2x-3)}$ This is our final answer.

Finding Domain Algebraically

For these next problems, we are looking for what x-values will make the function defined. Any x-value that makes the function undefined (division by zero, square root of a negative number) is not included in the domain. We will now look at several examples.

EXAMPLE: Find the domain: $y=2x-5$

There is no place where you can divide by zero or take the square root of a negative number, so the domain would be all reals, indicated by $(-\infty,\infty)$.

EXAMPLE: Find the domain: $y=5x^2-3$

There is no place where you can divide by zero or take the square root of a negative number, so the domain would be all reals, indicated by $(-\infty,\infty)$.

EXAMPLE: Find the domain: $y=\frac{x-3}{2x-5}$

Here it is possible to have a zero in the denominator. The denominator is not allowed to be zero, so solve: $2x-5\neq 0$. Solving this you will get $x\neq\frac{5}{2}$. This means any number but five halves will work. To write this in interval notation it would be: $\left(-\infty,\frac{5}{2}\right)\cup\left(\frac{5}{2},\infty\right)$. Notice we did not set the top equal to zero because it is okay to have a zero on top. As long as you are not dividing by zero then this is okay.

EXAMPLE: Find the domain: $y = \sqrt{2x-5}$

For this one you need to make sure you do not take the square root of a negative number. The only numbers that will work are positive numbers, so solve this equation: $2x - 5 \geq 0$. It is okay for our answer to equal 0. Solving it you will get $x \geq \frac{5}{2}$. In interval notation this would look like $\left[\frac{5}{2}, \infty\right)$.

EXAMPLE: Find the domain: $y = \frac{7x}{\sqrt{2x-5}}$

This has two domains restrictions. First the denominator can't be zero. Also we are not allowed to have negative numbers under the square root. We will set it up almost the same as before, but this time we will not include zero. We want to solve: $2x - 5 > 0$. We don't want a zero in the denominator, so we don't include it in our answer. Solving this we get $x > \frac{5}{2}$ and the interval notation would be $\left(\frac{5}{2}, \infty\right)$.

EXAMPLE: Find the domain: $y = \frac{x}{\sqrt[3]{x-3}}$

The bottom root has an odd index (little number next to radical, which is 3). This is not a square root. Since we have an odd index that means that if we take the odd root of a negative number, we will get something defined. There for the only domain restriction is if the bottom equals zero, so we solve $x - 3 \neq 0$, so $x \neq 3$. If we wanted to write this in interval notation it would be $(-\infty, 3) \cup (3, \infty)$.

EXAMPLE: Find the domain: $y = \frac{1}{x^2 + 9}$

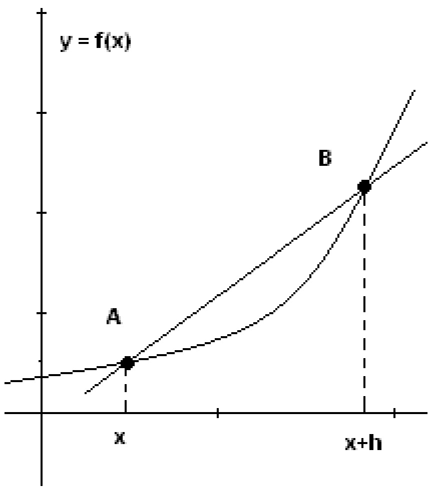
Since we have a fraction we need to set the denominator equal to zero. Let's look at the bottom for a second. Is it possible for us to get a zero on the bottom? The answer is no. If you try 3 as you would suspect is the answer it will not work since it will give you 18 since there is a plus sign. Since the bottom will never be zero that means we have no domain restrictions, so we can use any real number for x. Interval notation: $(-\infty, \infty)$.

EXAMPLE: Find the domain: $y = \frac{\sqrt{x-3}}{x-6}$

Since we have a fraction we need to set the denominator equal to zero. Solving will give us $x \neq 6$. On the top we have a square root, so we will solve the equation $x - 3 \geq 0$. We will get $x \geq 3$. Now we need to put both these statements together. It tells us that any number greater than and including 3 will work, except for 6. In interval notation it would be written as: $[3,\ 6) \cup (6,\ \infty)$.

More on next page…

Difference Quotient



If we wanted to find the slope of a curved line, the only way we can do this is by estimating it with a straight line. We will start with one point and then move over by a small amount h. Now we will use the slope formula. In the picture we have two points, A and B. The coordinates for these are: $(x, f(x))$ and $(x+h, f(x+h))$.

The slope, also called the **difference quotient** is: $\dfrac{f(x+h)-f(x)}{h}$

In calculus we will try to minimize h so that it is so small that we end up at a point, which will be the exact slope of the curved line at x.

Now let's look at some examples finding the difference quotient.

EXAMPLE: Let $f(x)=2x-3$. Find the difference quotient.

Let's first find $f(x+h)$. Once we have this we can put it into the difference quotient formula. Replace x in the original equation with x + h.

$f(x+h)=2(x+h)-3$ Now simplify.
$f(x+h)=2x+2h-3$

We are ready to substitute this into difference quotient formula. We have $f(x+h)$ and we also know $f(x)$, which is the original equation.

$\dfrac{2x+2h-3-(2x-3)}{h}$ Here we have substituted into the formula. Notice the parenthesis around $f(x)$.

$\dfrac{2x+2h-3-2x+3}{h}$ Now we distributed the minus sign and the last thing is to simplify.

$\dfrac{2h}{h}=2$ The 2x and the 3 canceled and then the h canceled, leaving us with our answer of 2.

Does 2 make sense? Yes, because a difference quotient tells you the slope at any value of x. Since we have a line the slope of 2 will not change no matter what value of x we use.

EXAMPLE: Let $f(x)=3x^2-x+1$. Find the difference quotient.

We will do this the same way as above. First we will find $f(x+h)$.

$f(x+h)=3(x+h)^2-(x+h)+1$ What is $(x+h)^2$? If you are thinking x^2+h^2 you are wrong. This is actually $(x+h)(x+h)$ which is a FOIL. It is $x^2+2xh+h^2$.

$f(x+h)=3(x+h)(x+h)-x-h+1$
$f(x+h)=3(x^2+2xh+h^2)-x-h+1$
$f(x+h)=3x^2+6xh+3h^2-x-h+1$

Now that we have simplified this as much as possible, we will put it into the difference quotient formula.

$\frac{3x^2+6xh+3h^2-x-h+1-(3x^2-x+1)}{h}$ Now we will distribute the minus into $f(x)$.

$\frac{3x^2+6xh+3h^2-x-h+1-3x^2+x-1}{h}$ Now cancel and simplify.

$\frac{6xh+3h^2-h}{h}$ Now we can factor out an h from the top.

$\frac{h(6x+3h-1)}{h}$ Last thing is we can cancel the h from top and bottom.

$6x+3h-1$ This is our answer. What can you do with this expression? Now if we were in calculus we would minimize the h (go to zero). What is left can be used to find the slope of this curve at any point x (derivative).

EXAMPLE: $f(x)=\frac{3}{x-1}$. Find the difference quotient.

$f(x+h)=\frac{3}{(x+h)-1}$ Nothing more we can do to simplify this. Now put it into the formula.

$\frac{\frac{3}{x+h-1}-\frac{3}{x-1}}{h}$ Now to simplify this we need add the fractions on the top. Need common denominators.

$\frac{\frac{(x-1)}{(x-1)}\frac{3}{\left(x+h-1\right)}-\frac{(x+h-1)}{(x+h-1)}\frac{3}{\left(x-1\right)}}{h}$ Now that we have common denominators, combine the fractions

$\frac{\frac{3(x-1)-3(x+h-1)}{(x-1)(x+h-1)}}{h}$ We can simplify the very top part of this fraction

$\frac{\frac{3x-3-3x-3h+3}{(x-1)(x+h-1)}}{h}$ Simplify.

$\frac{\frac{-3h}{(x-1)(x+h-1)}}{\frac{h}{1}}$ Now we will clear the double fractions by multiplying by the reciprocal.

$\frac{-3h}{h(x-1)(x+h-1)}$ Almost done. Just need to cancel out the h from the top and bottom.

$\frac{-3}{(x-1)(x+h-1)}$ Whew! Okay this is our answer.