4.6 The Real Zeros of Polynomial Functions

In the previous section, we learned how to divide polynomials. Now we will expand on that and learn some new theorems.

Remainder Theorem: If $f(x)$ is divided by x – c, then the remainder is $f(c)$
Factor Theorem: If $f(c)=0$ then x – c is a factor of $f(x)$. Also, if x – c is a factor of $f(x)$ then $f(c)=0$.

EXAMPLE: Given $f(x)=2x^3-4x^2+5x-3$, find $f(2)$. Then use the Remainder Theorem to find the remainder when $f(x)$ is divided by x – 2 . Finally use the Factor Theorem to determine whether $x-2$ is a factor of $f(x)$. Is $x = 2$ a zero of $f(x)$?

First let's find $f(2)$. From a previous section we find that $f(2)=2(2)^3-4(2)^2+5(2)-3$ which equals $f(2)=16-16+10-3=7$.

Now we will use synthetic division for this problem. I will just show the result of synthetic division here.

```
 2| 2  -4   5  -3
       4    0  10
   -------------
    2  0    5   7
```

The remainder is 7. Since this is not zero we know right away that $x-2$ is not a factor by the Factor Theorem. When we did synthetic division the number in the box was 2. Notice that this is the remainder that we got when we did synthetic division. This is what the Remainder Theorem says. This is an alternative way of evaluating the function at 2 without as many calculations. Since the remainder was 7 we know that $x = 2$ is not a zero of $f(x)$.

EXAMPLE: Given $f(x)=2x^4+12x^3+6x^2-5x+75$, find $f(-5)$. Then use the Remainder Theorem to find the remainder when $f(x)$ is divided by x + 5. Finally, use the Factor Theorem to determine whether x + 5 is a factor of $f(x)$. Is $x = -5$ a zero of $f(x)$?

First let's find $f(-5)$. From a previous section we find that $f(-5)=2(-5)^4+12(-5)^3+6(-5)^2-5(-5)+75$ which equals $f(-5)=2(625)+12(-125)+6(25)+25+75=0$.

Next we will use synthetic division:

```
-5| 2   12    6   -5    75
       -10  -10   20   -75
   -----------------------
    2    2   -4   15     0
```

This time we get a remainder of zero. This means that –5 is a zero of $f(x)$. The Factor Theorem tells us that $x + 5$ is a factor of $f(x)$ since it divided in evenly. The Remainder Theorem tells us that $f(-5)=0$. Notice again that the remainder when doing the synthetic division is the same as when we did $f(-5)$.

EXAMPLE: Solve the equation $2x^3 - 3x^2 - 11x + 6 = 0$ given that -2 is a zero of $f(x) = 2x^3 - 3x^2 - 11x + 6$.

We know that if we do synthetic division, we should get a remainder of zero:

-2	2	-3	-11	6
		-4	14	-6
	2	-7	3	0

We do get a zero for a remainder. This also tells us $2x^2 - 7x + 3$ is a factor of f(x). Because of this, we can factor $2x^2 - 7x + 3$ and set it equal to zero to find our other solutions. Factoring will give $(2x-1)(x-3) = 0$. We set both of these other factors equal to zero to get $x = \frac{1}{2}$ and $x = 3$.

Rational Zeros Theorem

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ be a polynomial. Then the number of possible real zeros of f is:

$$\frac{factors\ \ of\ \ a_0}{factors\ \ of\ \ a_n}$$

To review a factor is a number that evenly divides into something. For example, the factors of 6 are 1, 2 and 3.

EXAMPLE: List the possible real zeros of: $f(x) = 3x^3 - 7x^2 - 22x + 8$

Using the Rational Zeros Theorem we will write the factors of 8 over the factors of 3:

$\frac{\pm 1,\ \pm 2,\ \pm 4,\ \pm 8}{\pm 1,\ \pm 3}$ Now divide each number on top by each on the bottom. You will get the list of zeros:

$\pm 1,\ \pm 2,\ \pm 4,\ \pm 8,\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{8}{3}$. These do not need to be in any special order.

This list represents all the possible places the graph could cross the x-axis.

EXAMPLE: Given $f(x) = x^3 + 8x^2 + 11x - 20$, a.) Use the Rational Zero Theorem to find the list of possible zeros. b.) Find the zeros using synthetic division. c.) Use the zeros to factor the polynomial.

a.) We need to use the Rational Zeros Theorem to find our list of possible zeros.

$\frac{\pm 1,\ \pm 2,\ \pm 4,\ \pm 5,\ \pm 10,\ \pm 20}{\pm 1}$ Now divide each number on top by each on the bottom. You will get:

$\pm 1,\ \pm 2,\ \pm 4,\ \pm 5,\ \pm 10,\ \pm 20$. This is our list of possible zeros.

b.) Now we need to see which one is a zero. In order to do this, pick a zero to test and use synthetic division with the original equation. If we get a zero for the remainder we know that this is a zero. For example, let's first test the x-value -1. We need to set up and do the synthetic division:

```
-1 | 1   8   11  -20
         -1  -7   -4
    ----------------
     1   7    4  -24
```

We don't get a zero for the remainder, so we know -1 is not one of our answers. Let's try a different number. We will now test x = 1.

```
 1 | 1   8   11  -20
          1   9   20
    ----------------
     1   9   20    0
```

We do get a zero, so x = 1 is one of our answers. We could go through this process for all the other possible zeros in our list, but since we found one we can find the other ones in an easier way.

The last row of our synthetic division was 1 9 20. This means our new equation is $x^2 + 9x + 20$. We can set this equation equal to zero to get the other solutions. You will get $x^2 + 9x + 20 = 0$. Factoring this we will get $(x+4)(x+5) = 0$, so our other answers are x = -4 and x = -5. So our answer for this one would be: x = -5, -4, and 1.

c.) Now we need to use our zeros to factor this polynomial. After synthetic division with the first zero, we had $(x+4)(x+5) = 0$. This means that part of the factored answer is $(x+4)(x+5)$. Now we need the last piece, and this is formed with our first zero of $x = 1$. To form a factor, we need to use the formula $(x - c)$, where c is our factor. This means that (x – 1) is a factor. Therefore the factored form is: $f(x) = (x-1)(x+4)(x+5)$.

EXAMPLE: Given $f(x) = 3x^4 - 11x^3 - x^2 + 19x + 6$, a.) Use the Rational Zero Theorem to find the list of possible zeros. b.) Find the zeros using synthetic division. c.) Use the zeros to factor the polynomial.

a.) We need to use the Rational Zeros Theorem to find our list of possible zeros.

$\dfrac{\pm 1,\ \pm 2,\ \pm 3,\ \pm 6}{\pm 1,\ \pm 3}$ Now divide each number on top by each on the bottom. You will get:

$\pm 1,\ \pm 2,\ \pm 3,\ \pm 6,\ \pm\frac{1}{3},\ \pm\frac{2}{3}$. I did not put $\pm\frac{3}{3},\ \pm\frac{6}{3}$ because this is the same as $\pm 1,\ \pm 2$. We already have it.

b.) Now we need to see which one is a zero. We need to start testing zeros. We will start with x = -1 again.

```
-1 | 3  -11   -1   19   6
         -3   14  -13  -6
    ----------------------
     3  -14   13    6   0
```

We do get a zero here. This will leave us with an x cubed equation which we will not be able to factor very easily. We need to find another zero so that we can reduce this cube to a square.

```
 2 | 3  -11   -1   19   6
          6  -10  -22  -6
    ----------------------
     3   -5  -11   -3   0
```

After more trial and error we get x = 2 is another zero.

Let's look at the first row of numbers we got when we used x = -1. We got 3 -14 13 6 . Since we know x = -2 is also a zero, let's use synthetic division with this row and the second zero we found, x = -2:

```
 2 | 3  -14   13    6
          6  -16   -6
    -----------------
     3   -8   -3    0
```

This will leave us with $3x^2 - 8x - 3$ which we can set equal to zero to find the other answers. (We also could have done synthetic division with the other row above: 3 -5 -11 -3 and used x = -1. Does not matter which one you do).

Now we can solve $3x^2 - 8x - 3 = 0$. Factoring we will get: $(x-3)(3x+1) = 0$. Solving this we will get x = 3, x = -1/3. Putting this all together our final answer for part b will be x = -1, -1/3, 2, and 3.

c.) Now we need to use our zeros to factor this polynomial. After synthetic division with the first zero, we had $(x-3)(3x+1)=0$. This means that part of the factored answer is $(x-3)(3x+1)$. Now we need the last piece, and this is formed with our first zero of $x = 2$. To form a factor, we need to use the formula $(x - c)$, where c is our factor. This means that $(x - 2)$ is a factor. Therefore the factored form is: $f(x)=(x-2)(x-3)(3x+1)$.

EXAMPLE: Given $f(x)=x^4+4x^3+3x^2-4x-4$, a.) Use the Rational Zero Theorem to find the list of possible zeros. b.) Find the zeros using synthetic division. c.) Use the zeros to factor the polynomial.

a.) We need to use the Rational Zeros Theorem to find our list of possible zeros.

$\frac{\pm 1,\ \pm 2,\ \pm 4}{\pm 1}$ Now divide each number on top by each on the bottom. You will get: $\pm 1,\ \pm 2,\ \pm 4$.

b.) Now we need to see which one is a zero. We need to start testing zeros. We will start with x = -1 again.

```
-1| 1   4   3  -4  -4
       -1  -3   0   4
    ------------------
    1   3   0  -4   0
```

We do get a zero here. This will leave us with an x cubed equation which we will not be able to factor very easily. We need to find another zero so that we can reduce this cube to a square.

```
 1| 1   4   3  -4  -4
        1   5   8   4
    ------------------
    1   5   8   4   0
```

We find that x = 1 is another zero.

Let's look at the first row of numbers we got when we used x = -1. We got 1 3 0 -4 . Since we know x = 1 is also a zero, let's use synthetic division with this row and the second zero we found, x = 1:

```
 1| 1   3   0  -4
        1   4   4
    --------------
    1   4   4   0
```

This will leave us with x^2+4x+4 which we can set equal to zero to find the other answers. (We also could have done synthetic division with the other row above: 1 5 8 4 and used x = -1. Does not matter which one you do).

Now we can solve $x^2+4x+4=0$. Factoring we will get: $(x+2)(x+2)=0$. Solving this we will get x = -2. Putting this all together our final answer for part b will be $x=\pm 1,\ -2$. Technically, we do get a double root when we got x = -2, however we don't need to write the repeated zero.

c.) Now we need to use our zeros to factor this polynomial. After synthetic division with the first two zeros, we got $(x+2)(x+2)=0$. This means that part of the factored answer is $(x+2)(x+2)$, or $(x+2)^2$. Now we need the last pieces, and these is formed with our first zeros of $x = 1$ and $x = -1$. To form a factor, we need to use the formula $(x - c)$, where c is our factor. This means that $(x - 1)$ is a factor, and $(x + 1)$ is another factor. Therefore the factored form is: $f(x)=(x-1)(x+1)(x+2)^2$.