4.7 The Complex Zeros of Polynomial Functions

This section is similar to the last one except we will be finding complex, or imaginary zeros. Complex, or imaginary zeros always come in conjugate pairs. For example, if it is known that one zero is 7 + 3i, the other one must be 7 – 3i.

EXAMPLE: If a degree 6 polynomial has complex zeros i, 3 – 2i, and -2 + i, what must the other zeros be?

We just need to find the conjugate pairs. They would be –i, 3 + 2i, and -2 – i.

Now we will look at how to form a polynomial if imaginary zeros are given.

EXAMPLE: Form a degree 4 polynomial with real number coefficients if it has complex zeros of i and 1 + 2i.

Since the zeros come in conjugate pairs we know that –i and 1 – 2i are also zeros. If r is a zero then we know that x – r is a factor. So we will form the polynomial by taking x minus whatever the zero is:

$(x-i)(x-(-i))(x-(1+2i))(x-(1-2i))$	Now we simplify.
$(x-i)(x+i)(x-1-2i)(x-1+2i)$	I will multiply the first two together to begin with.
$(x^2+1)(x-1-2i)(x-1+2i)$	Now multiply the second two together.
$(x^2+1)(x^2-x+2xi-x+1-2i-2xi+2i-4i^2)$	I will cancel out like terms and make $i^2=-1$.
$(x^2+1)(x^2-2x+1-4(-1))$	Now simplify.
$(x^2+1)(x^2-2x+5)$	We have our polynomial.

EXAMPLE: The polynomial $g(x)=x^3+3x^2+25x+75$ has one zero of -5i. Use this to find the other zeros.

We know automatically that another zero is 5i. We want to form a polynomial like we did in the previous problem so we will have a factor of g.

$(x-5i)(x-(-5i))$	We set up the multiplication step.
$(x-5i)(x+5i)$	Now we multiply this together to get the polynomial.
x^2+25	So know we know that x^2+25 is a factor. This means if we divide this into f we should have a remainder of zero and we can find the other factor. We need to do long division on this one. Synthetic won't work since we are not dividing by a linear factor. The following work has been done on the long division problem:

$$\begin{array}{r} x+3 \\ x^2+0x+25\overline{)\,x^3+3x^2+25x+75} \\ \underline{x^3+0x^2+25x} \\ 3x^2+0x+75 \\ \underline{3x^2+0x+75} \\ 0 \end{array}$$

By doing this division we end up with x + 3. Since there is no remainder we know that x + 3 must be a factor of f. Since we know this is a factor we can set this factor equal to zero to find the final zero.

We set x + 3 = 0 and solve to get our final zero of x = -3. So our complete answer is $x=-3,\ \pm 5i$.

EXAMPLE: The polynomial $g(x) = x^4 - 7x^3 + 14x^2 - 38x - 60$ has one zero of 1 + 3i. Use this to find the other zeros.

We know automatically that another zero is 1 – 3i. We want to form a polynomial like we did in the previous problem so we will have a factor of g.

$(x-(1+3i))(x-(1-3i))$	We set up the multiplication step.
$(x-1-3i)(x-1+3i)$	Now we multiply this together to get the polynomial.
$x^2 - x + 3xi - x + 1 - 3i - 3xi + 3i - 9i^2$	Now simplify this and put in a -1 for i^2
$x^2 - 2x + 1 - 9(-1)$	
$x^2 - 2x + 10$	

We know we know that $x^2 - 2x + 10$ is a factor. This means if we divide this into f we should have a remainder of zero and we can find the other factor. We need to do long division on this one. The following work has been done on the long division problem:

$$\begin{array}{r} x^2 - 5x - 6 \\ x^2 - 2x + 10 \overline{\big) x^4 - 7x^3 + 14x^2 - 38x - 60} \\ \underline{x^4 - 2x^3 + 10x^2} \qquad\qquad \\ -5x^3 + 4x^2 - 38x \qquad \\ \underline{-5x^3 + 10x^2 - 50x} \qquad \\ -6x^2 + 12x - 60 \\ \underline{-6x^2 + 12x - 60} \\ 0 \end{array}$$

By doing this division we end up with x + 3. Since there is no remainder we know that x + 3 must be a factor of f. Since we know this is a factor we can set this factor equal to zero to find the final zero.

We set $x^2 - 5x - 6 = 0$ and solve to get our other zeros. Factoring will give you $(x+1)(x-6) = 0$, and we will get x = -1 and x = 6. We write our final answer as $x = -1,\ 6,\ 1 \pm 3i$.

EXAMPLE: The polynomial $g(x) = 6x^4 - 7x^3 + 93x^2 - 112x - 48$ has one zero of 4i. Use this to find the other zeros.

We know automatically that another zero is -4i. We want to form a polynomial like we did in the previous problem so we will have a factor of g.

$(x-4i)(x-(-4i))$	We set up the multiplication step.
$(x-4i)(x+4i)$	Now we multiply this together to get the polynomial.
$x^2 + 16$	

We know we know that $x^2 + 16$ is a factor. This means if we divide this into f we should have a remainder of zero and we can find the other factor. We need to do long division on this one. The following work has been done on the long division problem:

$$
\begin{array}{r}
6x^2 - 7x - 3 \\
x^2 + 0x + 16 \overline{\smash{\big)}\, 6x^4 - 7x^3 + 93x^2 - 112x - 48} \\
\underline{6x^4 + 0x^3 + 96x^2} \qquad\qquad\quad \\
-7x^3 - 3x^2 - 112x \qquad \\
\underline{-7x^3 + 0x^2 - 112x} \qquad \\
-3x^2 + 0x - 48 \\
\underline{-3x^2 + 0x - 48} \\
0
\end{array}
$$

By doing this division we end up with x + 3. Since there is no remainder we know that x + 3 must be a factor of f. Si Since we know this is a factor we can set this factor equal t to zero to find the final zero.

We set $6x^2 - 7x - 3 = 0$ and solve to get our other zeros. Factoring will give you $(3x+1)(2x-3) = 0$, and we will get $x = -\frac{1}{3},\ \frac{3}{2}$. We write our final answer as $x = -\frac{1}{3},\ \frac{3}{2},\ \pm 4i$

EXAMPLE: Given $f(-5) = 0$, find the complex zeros of $f(x) = x^3 + 13x^2 + 57x + 85$.

Since a zero is given we can use synthetic division like we did in the last section.

-5 |	1	13	57	85
		-5	-40	-85
	1	8	17	0

Since we started out with a cube, doing synthetic division brought this down to a quadratic. We now have to solve $x^2 + 8x + 17 = 0$. We see that it can't be factored. One way to solve it is by using the quadratic formula.

$$x = \frac{-8 \pm \sqrt{8^2 - 4(1)(17)}}{2(1)} = \frac{-8 \pm \sqrt{64 - 68}}{2} = \frac{-8 \pm \sqrt{-4}}{2} = \frac{-8 \pm 2i}{2} = -4 \pm i.$$

We write our answer as x = $-5,\ -4 \pm i$.

EXAMPLE: Given $f(x) = 2x^4 + x^3 - 35x^2 - 113x + 65$, a.) Use the Rational Zero Theorem to find the list of possible zeros. b.) Find the zeros using synthetic division given $f(5) = 0$ and $f\left(\frac{1}{2}\right) = 0$. c.) Use all the zeros to factor the polynomial.

a.) We need to use the Rational Zeros Theorem to find our list of possible zeros.

$\frac{\pm 1,\ \pm 5,\ \pm 13,\ \pm 65}{\pm 1,\ \pm 2}$ Dividing gives us $\pm 1,\ \pm 5,\ \pm 13,\ \pm 65,\ \pm\frac{1}{2},\ \pm\frac{5}{2},\ \pm\frac{13}{2},\ \pm\frac{65}{2}$.

b.) We were given the first two zeros so we don't need to do trial and error on this one. We will do synthetic division twice to get a quadratic. It doesn't matter which zero you start with. I will start with x = 5.

5 |	2	1	-35	-113	65
		10	55	100	-65
	2	11	20	-13	0

½	2	11	20	-13
		1	6	13
	2	12	26	0

Again it does not matter what zero you start with.
Since we started out with a fourth power, doing synthetic division twice brought this down to a quadratic. We now have to solve $2x^2 + 12x + 26 = 0$. We notice there is a common factor of 2. We can divide both sides of the equation by 2 to make our calculations easier. You will get $x^2 + 6x + 13 = 0$. This cannot be factored, so we must use the quadratic formula:

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(13)}}{2(1)} = \frac{-6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i .$$

We do not get a real number as a result, but we still need to include this in our answer.

We write our answer as x = $\frac{1}{2},\ 5,\ -3 \pm 2i$.

c.) Now we need to use our zeros to factor this polynomial. After synthetic division with the first two given zeros, we had $2x^2 + 12x + 26 = 0$. This did not factor, so this means that part of the factored answer is $2x^2 + 12x + 26$. Now we need the last pieces, and these are formed with our first zeros of x = 5 and x = 1/2. To form a factor, we need to use the formula ($x - c$), where c is our factor. This means that (x – 5) is a factor and (x – ½) is another factor. We can form the polynomial: $f(x) = \left(x - \frac{1}{2}\right)(x-5)(2x^2 + 12x + 26)$. But you can factor out a 2 from the last term and write it in front of everything: $f(x) = 2\left(x - \frac{1}{2}\right)(x-5)(x^2 + 6x + 13)$. Then you can multiply the 2 into the first term to get rid of the fraction: $f(x) = (2x-1)(x-5)(x^2 + 6x + 13)$.