5.5 Logarithmic Properties

Properties of Logarithms

1.) $\log_b 1 = 0$ Example: $\log_3 1 = 0$, $\ln 1 = 0$

2.) $\log_b b = 1$ Example: $\log_2 2 = 1$, $\ln e = 1$, $\log 10 = 1$

3.) $b^{\log_b M} = M$ Example; $2^{\log_2 5} = 5$, $5^{\log_5 \pi} = \pi$

4.) $\log_b b^r = r$ Example: $\log_3 3^7 = 7$, $\log_2 2^5$

5.) $\log_b M^r = r \cdot \log_b M$ Example: $\log_3 5^8 = 8 \cdot \log_3 5$

6.) $\log_b (M \cdot N) = \log_b M + \log_b N$ Example: $\log_2 (3 \cdot 5) = \log_2 3 + \log_2 5$

7.) $\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N$ Example: $\log_3 \left(\frac{24}{6}\right) = \log_3 24 - \log_3 6$

EXAMPLE: Find the exact value using logarithm properties: $\log_3 \sqrt[9]{3}$.

This can be written as $\log_3 3^{\frac{1}{9}}$ This uses the 4th property, so the answer is $\frac{1}{9}$.

EXAMPLE: Find the exact value using logarithm properties: $e^{\ln 6}$.

This one uses property #3, so the answer is 6.

EXAMPLE: Find the exact value using logarithm properties: $\log_2 \left[\log_{1/2} \left(\frac{1}{4} \right) \right]$.

First start with the inside log. We will write the ¼ as a power of ½: $\log_{1/2} \left(\frac{1}{2} \right)^2$. We can use property #4, and this will reduce to 2. So now the problem becomes $\log_2 2$. Property #2 tells us that our final answer will be 1.

EXAMPLE: Find the exact value using logarithm properties: $5^{\log_5 6 + \log_5 7}$.

First we will use property #6. This will give us: $5^{\log_5 (6 \cdot 7)}$. This equals $5^{\log_5 42}$. We will use property #3 to get our answer of 42.

EXAMPLE: Express $\log_9 x^2 \cdot \sqrt{3x-5}$ as a sum or difference of logarithms. Express powers as factors.

We will first use property #6 to break this apart. You will get $\log_9 x^2 + \log_9 \sqrt{3x-5}$. The square root can be written as a ½ power: $\log_9 x^2 + \log_9 (3x-5)^{\frac{1}{2}}$. Now use property #5 to bring the powers down in front of the logs since it wants us to express powers as factors: $2 \cdot \log_9 x + \frac{1}{2} \cdot \log_9 (3x-5)$. This is our answer.

EXAMPLE: Express $\ln \frac{(x+5)^4}{x^3}$ as a sum or difference of logarithms. Express powers as factors.

We will first use property #7 to break this apart. You will get $\ln(x+5)^4 - \ln x^3$. Now use property #5 to bring the powers down in front of the logs since it wants us to express powers as factors: $4 \cdot \ln(x+5) - 3\ln x$.

EXAMPLE: Express $\log_4 \frac{(x-5)^5 \cdot \sqrt[3]{x-2}}{(x-1)^4}$ as a sum or difference of logarithms. Express powers as factors.

We will first use property #7 to break this apart. You will get $\log_4 (x-5)^5 \cdot \sqrt[3]{x-2} - \log_4 (x-1)^4$. Now we can use property #6 to break up the first log. You will get: $\log_4 (x-5)^5 + \log_4 \sqrt[3]{x-2} - \log_4 (x-1)^4$. We can rewrite the cube root as a 1/3 power: $\log_4 (x-5)^5 + \log_4 (x-2)^{\frac{1}{3}} - \log_4 (x-1)^4$. Now use property #5 to bring the powers down in front of the logs since it wants us to express powers as factors:
$5\log_4 (x-5) + \frac{1}{3}\log_4 (x-2) - 4 \cdot \log_4 (x-1)$.

EXAMPLE: Express $\ln\left[\frac{x^2-5x+6}{(x+2)^3}\right]^{\frac{1}{4}}$ as a sum or difference of logarithms. Express powers as factors.

We will first use property #5 to bring down the ¼. You will get $\frac{1}{4} \cdot \ln\left[\frac{x^2-5x+6}{(x+2)^3}\right]$. Now factor the inside to see if anything can be canceled: $\frac{1}{4} \cdot \ln\left[\frac{(x-2)(x-3)}{(x+2)^3}\right]$. We will use property #6 and #7 to break apart this log:
$\frac{1}{4}\left[\ln(x-2) + \ln(x-3) - \ln(x+2)^3\right]$. Now we use property #5 again with that third term:
$\frac{1}{4}\left[\ln(x-2) + \ln(x-3) - 3\ln(x+2)\right]$. You can either leave your answer as this or you could distribute:
$\frac{1}{4}\ln(x-2) + \frac{1}{4}\ln(x-3) - \frac{3}{4}\ln(x+2)$.

EXAMPLE: Express $\log_5 \frac{2x^4(x-4)}{5(x-3)^5}$ as a sum or difference of logarithms. Express powers as factors.

When we use property #6 and 7 to break this apart, everything on top will have a positive sign in front of it and everything on the bottom will have a negative sign in front of it. After using the properties we will get: $\log_5 2 + \log_5 x^4 + \log_5(x-4) - \log_5 5 - \log_5(x-3)^5$. Notice that the term $2x^4$ can also be broken up into $2 \cdot x^4$. For the term $\log_5(x-3)^5$, we can use property #5 to bring down the power: $\log_5 2 + 4 \cdot \log_5 x + \log_5(x-4) - \log_5 5 - 5 \cdot \log_5(x-3)$. We can do one more thing with $\log_5 5$. We can use property #2 to get our final answer: $\log_5 2 + 4 \cdot \log_5 x + \log_5(x-4) - 1 - 5 \cdot \log_5(x-3)$. It doesn't matter the order in which you write the terms.

EXAMPLE: Express $\frac{1}{2} \cdot \log_6 x + 2 \cdot \log_6(x-2)$ as a single logarithm without negative exponents.

These problems are having us do the opposite steps of what we were doing in the previous powers. In the previous problems the last step we did was use property #5 to bring down the powers, so now this will be our first step. We will bring up the powers using property #5: $\log_6 x^{\frac{1}{2}} + \log_6(x-2)^2$. Now we will use property #6 to combine these together into one log: $\log_6 x^{\frac{1}{2}}(x-2)^2$, or $\log_6 \sqrt{x}(x-2)^2$.

EXAMPLE: Express $\ln(x^2-1) - 2 \cdot \ln(x+1)$ as a single logarithm without negative exponents.

First use property #5 to bring up the power on the second term: $\ln(x^2-1) - \ln(x+1)^2$. Now use property #7 to write these as a single log: $\ln \frac{x^2-1}{(x+1)^2}$. We can factor the numerator using the difference of squares: $\ln \frac{(x+1)(x-1)}{(x+1)^2}$. Now we can reduce to get our answer: $\ln \frac{x-1}{x+1}$.

EXAMPLE: Express $-2 \cdot \log_3 x + \log_3(x^2+2) - \log_3 4 + 3 \cdot \log_3 x + \log_3 8$ as a single logarithm without negative exponents.

I will rewrite this with the positive terms first and the negative terms following: $\log_3(x^2+2) + 3 \cdot \log_3 x + \log_3 8 - 2 \cdot \log_3 x - \log_3 4$. What I will do first is to factor out the negative from the last two terms: $\log_3(x^2+2) + 3 \cdot \log_3 x + \log_3 8 - (2 \cdot \log_3 x + \log_3 4)$. Now that the negative has been factored out I will use property #5 to bring up the powers: $\log_3(x^2+2) + \log_3 x^3 + \log_3 8 - (\log_3 x^2 + \log_3 4)$. I will now use property #6 to condense these: $\log_3 8x^3(x^2+2) - (\log_3 4x^2)$. Now we can use property #7 to write these as a single log: $\log_3 \frac{8x^3(x^2+2)}{4x^2}$. Finally we can reduce this: $\log_3 2x(x^2+2)$.

Change-of-Base Formula

In the last section we mentioned that the only type of logs we can do on our calculator is base 10 and base e. This formula will allow us to find the numerical value of a log with ANY base.

$\log_a M = \dfrac{\log_b M}{\log_b a}$ In this formula the b can be any base we want.

EXAMPLE: Use the Change-of-Base Formula to calculate $\log_5 8$ to two decimal places.

In this problem M = 8 and a = 5. Since we want to do this on the calculator we want b to be either 10 or e since these are the only bases we can do on our calculator: $\log_5 8 = \dfrac{\log 8}{\log 5} \approx 1.29$. Notice we are using base 10 here. You could also have done: $\log_5 8 = \dfrac{\ln 8}{\ln 5} \approx 1.29$. Both will give you the same answer.

EXAMPLE: Use the Change-of-Base Formula to calculate $\log_3 8 \cdot \log_8 9$ to two decimal places.

We can rewrite this using the formula twice: $\log_3 8 \cdot \log_8 9 = \dfrac{\log 8}{\log 3} \cdot \dfrac{\log 9}{\log 8}$. (You could have also used natural logs here (ln). We can cancel the $\log 8$ from the top and bottom to get $\dfrac{\log 9}{\log 3}$. Notice $\dfrac{\log 9}{\log 3} \neq \log 3$. You cannot divide 9 by 3 since it is inside the log. We calculate log 9 and log 3 separately and divide to get our answer of 2.