5.7 Financial Models

Compound Interest

The amount *A* after *t* years due to a principle *P* invested at an annual interest rate *r* (expressed as a decimal) compounded n times per year is $A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$. The compound interest formula actually starts with the simple interest formula, $I = Prt$. Interest is calculated once, and then this interest is added to the principle, and the formula is repeated again. This process continues until it is compounded *n* times. So instead of that long process, we have the compound interest formula.

In working with these kinds of problems, they will give you different payment periods as listed below, which also tell you what to enter for *n*:

Annually: Once a year, $n = 1$
Semiannually: Twice a year, $n = 2$
Quarterly: Four times a year, $n = 4$
Monthly: 12 times a year, $n = 12$
Weekly: 52 times a year, $n = 52$
Daily: 365 days per year, $n = 365$

EXAMPLE: $300 is invested at 12% compounded monthly for $1\frac{1}{2}$ years. Find the amount that results from this investment.

Here $n = 12$ since it is compounded monthly. The principle is P = $300, and $r = 0.12$ and $t = 1.5$. We will put these into the compound interest formula: $A = 300 \cdot \left(1 + \frac{0.12}{12}\right)^{12 \cdot (1.5)}$. First simplify inside the parenthesis and also multiply the *n* and *t* together in the exponent position: $A = 300 \cdot (1.01)^{18}$. Now we will raise 1.01 to the power of 18. You will get $A = 300 \cdot 1.196147...$ Now multiply to get our answer: $A = \$358.84$.

EXAMPLE: If a person borrows $5100 and, after 3 months, pays off the loan in the amount of $5227.50, what per annum rate of interest was charged?

If a principle of *P* dollars is borrowed for a period of *t* years at a per annum interest rate *r*, expressed as a decimal, then interest I charged is $I = P \cdot r \cdot t$. The future value is the sum of the principle and interest. The formula we want to use is $A = P + P \cdot r \cdot t$ or $A = P(1 + rt)$. The t must be in years, so we need to convert 3 months into years: $t = 3/12 = 0.25$. The principle is $5100 and the amount is $5227.50. Now solve:

$$A = P(1 + rt)$$
$$5227.50 = 5100(1 + 0.25r)$$
$$5227.50 = 5100 + 1275r$$
$$127.50 = 1275r$$
$$r = \frac{127.50}{1275} = 0.10 = 10\%.$$

Continuous Compounding

If you take the same compound interest formula as above and have n go to infinity, you will get $A = P \cdot e^{rt}$, which is the formula for compounding continuously.

EXAMPLE: \$300 is invested at 12% compounded continuously for $1\frac{1}{2}$ years. Find the amount that results from this investment.

Since we are compounding continuously, we will use $A = P \cdot e^{rt}$, which P = \$300, r = 0.12, and t = 1.5. Plug it in to get: $A = 300 \cdot e^{0.12 \cdot (1.5)}$. Work the exponent part first: $A = 300 \cdot e^{0.18}$. Now we will raise e to the power of 0.18. Use the e^ key on your calculator: $A = 300 \cdot (1.197217...)$. So A = \$359.17. Notice that this yields a slightly higher investment when compared to compounding monthly.

EXAMPLE: Determine the rate that represents the better deal:

9% compounded quarterly or $9\frac{1}{4}\%$ compounded annually

For this problem, they do not give you a principle or time. To make it easy, let's assume P = 1000 and t = 1. Then we will apply the compound interest formula for each one separately. Note that we can use any value for P and t as long as the same ones are used for each rate. Now let's calculate:

9% compounded quarterly (n = 4, r = 0.09) $\quad A = 1000 \cdot \left(1 + \frac{0.09}{4}\right)^{4(1)} = 1000(1.0225)^4 = \$1093.08$

$9\frac{1}{4}\%$ compounded annually (n = 1, r = 0.0925) $\quad A = 1000 \cdot \left(1 + \frac{0.0925}{1}\right)^{1(1)} = 1000(1.0925) = \$1092.50$

The first option (9% compounded quarterly) is the better deal. It may not seem like much savings with such a small principle, however if you have a principle of, say, a million dollars, then the savings become more significant.

Effective Rate of Interest

Suppose that you have a certain amount to invest and a bank offers to pay you a certain percent annual interest compounded monthly. What simple interest rate is needed to earn an equal amount in one year? This is called the effective rate of interest.

The effective rate of interest, r_E, of an investment earning an annual interest rate, r, compounding n times per year is given by $r_E = \left(1 + \frac{r}{n}\right)^n - 1$.

EXAMPLE: Find the effective rate of interest for 24% compounded quarterly.

We will use the effective rate of interest formula. Here, n = 4 and r = 0.24. Plug these into the formula:

$$r_E = \left(1 + \frac{r}{n}\right)^n - 1 = \left(1 + \frac{0.24}{4}\right)^4 - 1 = \left(1 + \frac{0.24}{4}\right)^4 - 1 = 1.26248... - 1 = 0.26248 \text{ or } 26.248\%.$$

EXAMPLE: What rate of interest compounded annually is required to double an investment in 9 years?

If P is the principle initially invested, then after 9 years P will double. That is, the amount due, A, will be equal to $2P$. To find the interest rate required to double P, use the compound interest formula with $A = 2P$, $n = 1$, and $t = 9$ years, and solve for r.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$2P = P\left(1 + \frac{r}{1}\right)^{1 \cdot 9}$$
$$2 = (1 + r)^9$$
$$\sqrt[9]{2} = \sqrt[9]{(1 + r)^9}$$
$$\sqrt[9]{2} = 1 + r$$
$$\sqrt[9]{2} - 1 = r$$
$$r \approx 0.0801 \quad \text{or } 8.01\%$$

EXAMPLE: What will a \$90,000 house cost 10 years from now if the price appreciation for homes over that period averages 4% compounded annually?

We will use the compound interest formula with $P = 90000$, $t = 10$, $n = 1$, and $r = 0.04$:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$A = 90000\left(1 + \frac{0.04}{1}\right)^{1 \cdot 10}$$
$$A = 90000(1 + 0.04)^{10}$$
$$A = 90000(1.04)^{10}$$
$$A \approx \$133,221.99$$