

5.8 Exponential Growth and Decay; Newton's Law of Cooling

There are many applications to exponential functions that we will focus on in this section. First let's look at the exponential model.

Exponential Growth / Decay Model

$$A(t) = A_0 e^{k \cdot t}$$

$A(t)$ = current population, A_0 = original population, k = growth or decay constant
 t = time measured in any unit.

EXAMPLE: At the start of the experiment there are 125 cells. Three hours later there are 235 cells.

a.) What is the exponential growth formula? b.) How many cells are present after 5 hours? c.) How long (rounded to the nearest hour) will it take the population to reach 442 cells?

We are going to use the exponential growth model above. First we need to identify what information we are given. Since there are 125 cells at the start of the experiment we know $A_0 = 125$. It says "after three hours", so we know that $t = 3$. It says that after 3 hours there are 235 cells, so $A(t) = 235$ since this is a current population. The only thing we don't know is the growth constant k . We need to find this. First we will substitute the information we identified.

Part A.

$$235 = 125e^{3k}$$

We have substituted the information so now we need to solve for k .

$$\frac{235}{125} = \frac{125e^{3k}}{125}$$

First divide both sides by 125.

$$1.88 = e^{3k}$$

We need to get rid of the e so we will take the natural log of both sides.

$$\ln 1.88 = \ln e^{3k}$$

The $\ln e$ from the right side will cancel leaving us with $3k$.

$$\ln 1.88 = 3k$$

Now divide both sides by 3.

$$k = \frac{\ln 1.88}{3} \approx 0.2104$$

You can always round k to 4 decimal places.

Note, if you decide to use the whole number for k instead of rounding, this is okay. If you don't round anything then you might get slightly different answers than I got, but you will be close enough. You will still be correct if you are within a few decimal places. We are ready to answer part a. This asks us for the growth function.

When we write this we DO NOT put anything in for t or $A(t)$. Our answer to part a is $A(t) = 125e^{0.2104t}$

Now for part B. It asks for how many cells are present after 5 hours. We know that $t = 5$. We will put this into our growth formula: $A(5) = 125e^{0.2104(5)}$. To solve this we need to first multiply the two numbers in the exponent: $A(5) = 125e^{1.052}$. To solve this we need to find out what $e^{1.052}$ is. If you have a graphing calculator, hit the second key and then the \ln key to get e^\wedge . Then enter 1.052. If you have a scientific calculator you will need to enter 1.052 first and then hit the second key and then \ln . Either way you should get 2.8633721. Now enter this into our formula: $A(t) = 125(2.8633721) \approx 358$ cells. You can round it to the nearest cell.

Now for Part C. The 442 is the $A(t)$. We know everything except the t .

$$442 = 125e^{0.2104t} \quad \text{First divide both sides by 125.}$$

$$3.536 = e^{0.2104t} \quad \text{Take the natural log of both sides.}$$

$$\ln 3.536 = \ln e^{0.2104t} \quad \text{The } \ln e \text{ cancels, leaving you with } 0.2104t.$$

$$\ln 3.536 = 0.2104t \quad \text{Divide both sides by } 0.2104$$

$$t \approx 6 \text{ hours.}$$

EXAMPLE: At the start of the experiment there are 1000 bacteria. After 4 hours the population doubled. What is the exponential growth formula? How many bacteria are present after 6 hours?

Since the population doubled, we know that the new population will be 2000, therefore we know $A(t) = 2000$. The time is NOT 6 hours. We need to use the time it took the population to double, which is 4 hours, so $t = 4$. At the start of the experiment there are 1000 bacteria, so $A_0 = 1000$. We need to put this into the formula and solve for k :

$$2000 = 1000e^{4k} \quad \text{First divide both sides by 1000.}$$

$$2 = e^{4k} \quad \text{Now take the natural log of both sides.}$$

$$\ln 2 = \ln e^{4k} \quad \text{The } \ln e \text{ will cancel, leaving us with } 4k \text{ on the right side.}$$

$$\ln 2 = 4k \quad \text{Divide both sides by 4 and round to 4 decimal places.}$$

$$k \approx 0.1733$$

Now we need to know the growth formula for this problem. It is: $A(t) = 1000e^{0.1733t}$. We need to know the population after 6 hours, so $t = 6$. We will put this into our growth formula: $A(t) = 1000e^{0.1733(6)}$. To solve this we need to first multiply the two numbers in the exponent: $A(t) = 1000e^{1.0398}$. To solve this we need to find out what $e^{1.0398}$ is first by using our calculator. You will get 2.8286512. Now enter this into our formula: $A(t) = 1000(2.8286512) \approx 2829$ bacteria.

EXAMPLE: Complete the table:

2000 pop (in millions)	Projected 2011 Pop (millions)	Projected Growth Rate
53.4		0.0108

Let's look at the table and define some given variables. Since we are going from 2000 to 2011, we know that $t = 11$. We are given the growth rate, so we know $k = 0.0108$. We also know the initial population in 2000, so $A_0 = 53.4$. We just need to find the final population. Let's plug in our given values into the growth formula and use our calculator: $A(t) = 53.4e^{0.0108(11)} = 53.4e^{0.1188} = 53.4(1.126...) = 60.1$ million.

Half-life Problems

The next problems will focus on types of decay. We will first look at half-life problems. Half-life is the amount of time it takes half of a substance to decay. The problems we will do will contain this information. We will still use the same exponential model, except that half-life problems have a special formula to solve for k .

Decay constant for a half-life problem: $k = \frac{-\ln 2}{\text{half life}}$.

EXAMPLE: The half-life of strontium-90 is 28 years. What is the decay function? How much of a 10 gram sample is left after 11 years?

We will not use the 90 in this problem. This is just the name of the isotope. We are given the half-life is 28 years, so we can put this into the formula to find our k : $k = \frac{-\ln 2}{28} \approx -0.0248$. Our k value should be negative since we are working with a decay problem. We know that the initial amount is 10 grams. Here is our decay function: $A(t) = 10e^{-0.0248t}$. Notice whenever we find the growth function we do not put anything in for t or $A(t)$.

To answer the second question, we will now put in a 11 for t : $A(t) = 10e^{-0.0248(11)}$. First multiply the two numbers in the exponent: $A(t) = 10e^{-0.2728}$. Then we do the e part on our calculator and you should now have: $A(t) = 10(0.76124502) = 7.61 \text{ grams}$. You can round your answer to 2 decimal places.

EXAMPLE: The half-life of radium-226 is 1620 years. How much of a 2 gram sample is left after 1000 years?

We will not use the 226 in this problem. This is just the name of the isotope. We are given the half-life is 1620 years, so we can put this into the formula to find our k : $k = \frac{-\ln 2}{1620} \approx -4.27868E - 4$. Your scientific calculator may display what I put above. If it has an E next to it then this means the calculator is displaying the answer in scientific notation. You need to move the decimal 4 places to the left, so $k = -0.0004$. We know that the initial amount is 2 grams. Here is our decay function: $A(t) = 2e^{-0.0004t}$.

To answer the second question, we will now put in a 1000 for t : $A(t) = 2e^{-0.0004(1000)}$. First multiply the two numbers in the exponent: $A(t) = 2e^{-0.4}$. Then we do the e part on our calculator and you should now have: $A(t) = 2(0.67032004) = 1.34 \text{ grams}$.

Other types of decay problems include Newton's Law of Cooling...

Newton's Law of Cooling:

$$u(t) = T + (u_o - T)e^{kt}, \text{ where } k < 0.$$

$u(t)$ = final temperature after cooling

T = temperature of the surrounding atmosphere

u_o = initial temperature before cooling

k = cooling constant (must be negative)

t = time

EXAMPLE: A pizza removed from the oven has a temperature of $450^\circ F$. It is left sitting in a room that has a temperature of $70^\circ F$. After 5 minutes the temperature of the pizza is $300^\circ F$. What is the temperature of the pizza after 20 minutes? When will the temperature of the pizza be $140^\circ F$?

Let's first define our variables. The initial temperature, u_o is $450^\circ F$. The temperature of the room is the atmosphere's temperature, so $T = 70^\circ F$. The time, t , is 5 minutes. The final temperature after cooling is $300^\circ F$, so this is $u(t)$. We have everything we need to solve for the cooling constant, k .

$$300 = 70 + (450 - 70)e^{5k}$$

We substitute into the formula.

$$300 = 70 + 380e^{5k}$$

We simplify inside the parenthesis. Now subtract 70 from both sides.

$$230 = 380e^{5k}$$

Divide both sides by 380.

$$0.6053 = e^{5k}$$

Take the natural log of both sides

$$\ln 0.6052\dots = \ln e^{5k}$$

The $\ln e$ will cancel.

$$\ln 0.6052\dots = 5k$$

Divide both sides by 5.

$$k \approx -0.1004$$

We have our k value rounded to 4 decimal places.

Our cooling formula is now: $u(t) = 70 + (450 - 70)e^{-0.1004t}$, or $u(t) = 70 + 380e^{-0.1004t}$. Now to answer the first question we need to put in 20 for t . You will get:

$$u(20) = 70 + 380e^{-0.1004(20)}$$

First simplify multiply in the exponent.

$$u(20) = 70 + 380e^{-2.008}$$

Now calculate $e^{-2.008}$.

$$u(20) = 70 + 380(.13420755)$$

Now simplify to get the answer.

$$u(20) \approx 121^\circ F$$

To answer the second question, now we are given the final temperature after cooling, $u(t)$, which is $140^\circ F$.

$$140 = 70 + 380e^{-0.1004t}$$

This time we need to solve for t . Subtract 70 from both sides.

$$70 = 380e^{-0.1004t}$$

Divide both sides by 380.

$$0.1842 = e^{-0.1004t}$$

Take the natural log of both sides.

$$\ln 0.1842 = \ln e^{-0.1004t}$$

The $\ln e$ will cancel.

$$\ln 0.1842 = -0.1004t$$

Divide both sides by -0.1004.

$$t \approx 17 \text{ min}$$

EXAMPLE: A thermometer reading $72^\circ F$ is placed in a refrigerator where the temperature is a constant $38^\circ F$. After 2 minutes the thermometer reads $60^\circ F$. What will it read after 7 minutes? How long will it take before thermometer reads $39^\circ F$?

Let's first define our variables. The initial temperature, u_0 is $72^\circ F$. The temperature of the refrigerator is the atmosphere's temperature, so $T = 38^\circ F$. The time, t , is 2 minutes. The final temperature after cooling is $60^\circ F$, so this is $u(t)$. We have everything we need to solve for the cooling constant, k .

$$60 = 38 + (72 - 38)e^{2k}$$

We substitute into the formula.

$$60 = 38 + 34e^{2k}$$

We simplify inside the parenthesis. Now subtract 38 from both sides.

$$22 = 34e^{5k}$$

Divide both sides by 34.

$$0.6471 = e^{2k}$$

Take the natural log of both sides

$$\ln 0.6471 = \ln e^{2k}$$

The $\ln e$ will cancel.

$$\ln 0.6471 = 2k$$

Divide both sides by 2.

$$k \approx -0.2177$$

We have our k value rounded to 4 decimal places.

Our cooling formula is now: $u(t) = 38 + (72 - 38)e^{-0.2177t}$, or $u(t) = 38 + 34e^{-0.2177t}$. Now to answer the first question we need to put in 7 for t . You will get:

$$u(7) = 38 + 34e^{-0.2177(7)}$$

First simplify multiply in the exponent.

$$u(7) = 38 + 34e^{-1.5236}$$

Now calculate $e^{-1.5236}$.

$$u(7) = 38 + 34(0.21792305)$$

Now simplify to get the answer.

$$u(7) \approx 45.41^\circ F$$

To answer the second question, now we are given the final temperature after cooling, $u(t)$, which is $39^\circ F$.

$$39 = 38 + 34e^{-0.2177t}$$

This time we need to solve for t . Subtract 38 from both sides.

$$1 = 34e^{-0.2177t}$$

Divide both sides by 34.

$$0.0294 = e^{-0.2177t}$$

Take the natural log of both sides.

$$\ln 0.0294 = \ln e^{-0.2177t}$$

The $\ln e$ will cancel.

$$\ln 0.0294 = -0.2177t$$

Divide both sides by -0.2177 .

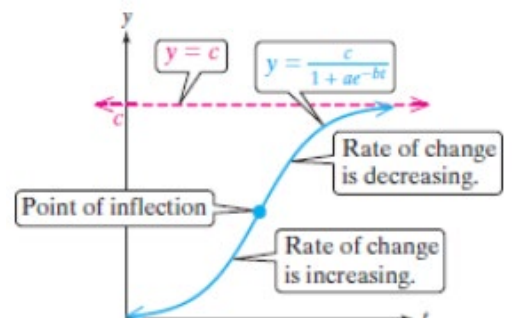
$$t \approx 16.2 \text{ min}$$

Logistic Growth Model

If you had a Petri dish with bacteria inside, the bacterial would grow exponentially, but not forever. Once all the food is used up the growth will level off. So a logistic growth model (listed below) best describes this situation. Use this model when there is a restriction to growth.

$$y = \frac{c}{1 + ae^{-bt}} \quad \text{where } a, b, c \text{ are constants.}$$

As in the figure, the c value indicates what the growth will level off at c . (horizontal asymptote). The c is also called the **limiting factor**. The term containing a and b will go to zero as t approaches infinity. The b term is the growth rate.



EXAMPLE: The population of Canada $P(t)$ (in millions) since January 1, 1900, can be approximated by

$$P(t) = \frac{55.1}{1 + 9.6e^{-0.02515t}} \text{ where } t \text{ is the number of years since January 1, 1900.}$$

a.) Evaluate $P(0)$ and evaluate its meaning.

To find $P(0)$, we put in a 0 for t : $P(0) = \frac{55.1}{1 + 9.6e^{-0.02515(0)}} = \frac{55.1}{1 + 9.6e^0} = \frac{55.1}{1 + 9.6(1)} \approx 5.2$. This means that on January 1, 1900, the Canadian population was about 5.2 million.

b.) What is the growth rate of the Canadian population?

The growth rate is the b value, which in this problem is 2.525%.

c.) Use the function to approximate the Canadian population on January 1, 2015. Round to the nearest tenth of a million.

We are not going to use 2015 for t . Remember t is the number of years since 1900, so $2015 - 1900 = 115$. To find $P(115)$, we put in a 115 for t :

$P(115) = \frac{55.1}{1 + 9.6e^{-0.02515(115)}} = \frac{55.1}{1 + 9.6e^{-2.89225}} = \frac{55.1}{1 + 9.6(0.05545\dots)} = \frac{55.1}{1 + 0.53233\dots} \approx 36$. This means that on January 1, 2015, the Canadian population was about 36 million.

d.) From the model, during which year would the Canadian population reach 45 million?

For this problem we will put in a 45 for $P(t)$ since they are now giving us a population and we need to solve for t :

$45 = \frac{55.1}{1 + 9.6e^{-0.02515t}}$	First cross multiply.
$45(1 + 9.6e^{-0.02515t}) = 55.1$	Now divide both sides by 45.
$1 + 9.6e^{-0.02515t} = 1.2244\dots$	Subtract 1 from both sides.
$9.6e^{-0.02515t} = 0.2244\dots$	Divide both sides by 9.6
$e^{-0.02515t} = 0.0233796\dots$	Take the natural log of both sides.
$\ln e^{-0.02515t} = \ln(0.0233796\dots)$	Simplify.
$-0.02515t = -3.75589\dots$	Divide.
$t \approx 149$	

So our answer is not 149. It asked for the year, so we need to add 149 to 1900 to get our final answer 2049.