

R.1 Solving Quadratic Equations

Quadratic Functions – Standard Form is $f(x) = ax^2 + bx + c$ where $a \neq 0$.

In this section we will look at a couple ways of finding the x-intercepts of a quadratic equation as defined above. We will solve them by factoring, square root method, and the quadratic equation. The book covers completing the square, but we will not focus on that method.

Factoring

EXAMPLE: Find the x-intercept by factoring: $f(x) = x^2 + 7x + 6$

Since we are find the x-intercepts the $f(x)$ will be zero. So we are trying to solve: $0 = x^2 + 7x + 6$. For quadratics in this form with a 1 in front of the squared term we need to find two numbers that multiply to make the last term but add to make the middle term. So we need to find two numbers that multiply to make 6 but add together to make 7. The numbers would be 1 and 6. So we factor: $0 = (x + 1)(x + 6)$. Setting both of these factors equal to zero you will get $x = -1$ and $x = -6$. As intercepts we write $(-1, 0)$ and $(-6, 0)$.

EXAMPLE: Find the x-intercept by factoring: $f(x) = 3x^2 - 6x - 24$

Since we are find the x-intercepts the $f(x)$ will be zero. So we are trying to solve: $0 = 3x^2 - 6x - 24$. Now it may seem like we have large numbers, but we should always see if we can factor out any common factors. In this problem all three numbers are divisible by 3, so 3 is a common factor. Factor this out and you will get $0 = 3(x^2 - 2x - 8)$. You can divide both sides by 3 to get: $0 = x^2 - 2x - 8$. Now we look for factors of -8 that add up to -2. These will be 2 and -4. We will get $0 = (x + 2)(x - 4)$. Setting both equal to zero we will get two answers: $x = -2$ and $x = 4$. As intercepts we write $(-2, 0)$ and $(4, 0)$.

EXAMPLE: Factor: $f(x) = 6x^2 - x - 12$

For this problem you have three different ways to solve it, but I will only show two ways. The third method is trial and error, but if a and c have lots of factors it could take a long time to factor it, which is why I will only focus on the first two methods.

I will solve the same problem two different ways.

Grouping Method

Multiply the a coefficient (6) by the constant c (-12). You will get -72. Now we need to find a way of rewriting the middle term b (-1) as factors of -72. Instead of $-x$ we will write $8x - 9x$. So now our problem becomes:

$6x^2 + 8x - 9x - 12$ From here we will factor by grouping (factor first two and last two terms separately).

$2x(3x + 4) - 3(3x + 4)$ Now factor out the common factor of $3x + 4$.

$(3x + 4)(2x - 3)$

Bottom's Up (Hervey) Method

This method you probably have not seen before. This method only works once you have taken out any common factors. You will still do the same step as in the grouping method. Multiply the first number by the last: $6(-12) = -72$. What will also happen is that the first number (6) disappears, and the -12 will turn into our product of -72. Here's what it will look like:

$$\begin{array}{ll}
 x^2 - x - 72 & \text{First number is gone and the -12 is now a -72. From here, factor normally.} \\
 (x+8)(x-9) & \text{Now the number that disappeared (6) is now written below each number in the parenthesis} \\
 \left(x + \frac{8}{6}\right)\left(x - \frac{9}{6}\right) & \text{So now we formed fractions. These ALWAYS need to be reduced in order to be correct.} \\
 \left(x + \frac{4}{3}\right)\left(x - \frac{3}{2}\right) & \text{Now that they are reduced, bring the "bottoms up". Bring the 3 in front of the x and bring the} \\
 (3x+4)(2x-3) & \text{2 in front of the x. This is where this process gets its name because the bottom comes up.} \\
 & \text{This is our final answer, same as with grouping.}
 \end{array}$$

EXAMPLE: Solve for x: $4x^2 - 38x + 70 = 0$

Notice that numbers are large on this one. We must see if there is a common factor first. We find that each number is divisible by 2, so we can factor out a 2: $2(2x^2 - 19x + 35) = 0$ Now we can factor what is inside the parenthesis. This doesn't ask for an intercept, but just to solve for x.

Grouping:

$2(35) = 70$. Want to find 2 factors of 70 that add to -19 so we can rewrite the middle term. The numbers that will work are -5 and -14. Now we rewrite the problem and finish by using factoring by grouping.

$$\begin{array}{l}
 2(2x^2 - 5x - 14x + 35) = 0 \\
 2(x(2x - 5) - 7(2x - 5)) = 0 \\
 2(2x - 5)(x - 7) = 0 \quad \text{So now we set each part equal to zero and solve.} \\
 x = \frac{5}{2} \text{ and } x = 7. \quad \text{What about the 2 in front? This can't be zero, so this does not give us another answer.}
 \end{array}$$

Bottom's Up (Hervey) Method:

$2(35) = 70$. Remember the first number (2) disappears and 35 is replaced by 70.

$$\begin{array}{l}
 2(x^2 - 19x + 70) = 0. \quad \text{Now factor as normal.} \\
 2(x - 5)(x - 14) = 0 \quad \text{Now divide both numbers by the number that disappeared (8).} \\
 2\left(x - \frac{5}{2}\right)\left(x - \frac{14}{2}\right) = 0 \quad \text{Reduce any fractions, if possible. We can only reduce the second fraction.} \\
 2\left(x - \frac{5}{2}\right)(x - 7) = 0 \quad \text{We can actually set this to zero now without bringing the bottom up.} \\
 2(2x - 5)(x - 7) = 0 \quad \text{If you did bring the bottom up you would have the same equation as above. Now solve.} \\
 x = \frac{5}{2} \text{ and } x = 7.
 \end{array}$$

EXAMPLE: Solve by factoring: $4x^2 = 24x$

A common mistake is to divide both sides by x . You are not allowed to do this because it is possible you will be dividing by zero. Instead, set this equation to zero by subtracting one of the terms: $4x^2 - 24x = 0$. Now factor out the common factor: $4x(x - 6) = 0$. Set both pieces equal to zero to get $x = 0$ and 6 .

EXAMPLE: Solve by factoring: $(x + 3)(x + 4) = 6$

A common mistake is to set each factor equal to 6 . This method only works if there is a zero due to the zero product property. The correct way to do this is to first expand the right side: $x^2 + 7x + 12 = 6$. Now subtract 6 from both sides to set it equal to zero: $x^2 + 7x + 6 = 0$. Now factor: $(x + 1)(x + 6) = 0$. Setting it equal to zero gives us two answers; $x = -1$ and -6 .

Square Root Method

This method involves taking the square root of both sides of the equation. Let's look at the above problem. We had $0 = x^2 - 36$. Instead of factoring we can just solve for x . We can add 36 to both sides to get $36 = x^2$. Then we can take the square root of both sides to get $\pm 6 = x$. Then we write it as $(-6, 0)$ and $(6, 0)$.

EXAMPLE: Solve $0 = (3x - 2)^2 - 4$ by using the square root method.

We could distribute and multiply here, but there is an easier way. We will solve this similarly to the problem above. First add 4 to both sides. You will get: $4 = (3x - 2)^2$. Now take the square root of both sides. $\pm 2 = 3x - 2$. Don't forget the plus and minus signs whenever you take the square root. Since we get two numbers we will solve two equations: $2 = 3x - 2$ and $-2 = 3x - 2$. Solve each one individually and we will get $x = \frac{4}{3}$ and $x = 0$. This is how we can leave our answer since we are not finding intercepts.

Completing the Square

If you cannot factor the quadratic and if it is not in the form in which you can use the square root method, then other methods must be used. One method is the completing the square method.

EXAMPLE: Solve $x^2 - 6x + 7 = 0$ by completing the square.

The idea of completing the square is to put into a form where we can apply the square root principle. First we will move the 7 to the other side: $x^2 - 6x = -7$. Completing the square is done with the middle coefficient. We need to take -6 , divide by 2 and then square the result. Here are the two steps: 1.) $\frac{-6}{2} = -3$ 2.) $(-3)^2 = 9$.

Now we add 9 to both sides of the equation: $x^2 - 6x + 9 = -7 + 9$. From here we can factor the right side: $(x - 3)^2 = 2$. Now take the square root of both sides: $\sqrt{(x - 3)^2} = \pm\sqrt{2}$. This simplifies to: $x - 3 = \pm\sqrt{2}$. Add 3 to both sides to get the answer: $x = 3 \pm \sqrt{2}$.

EXAMPLE: Solve $x^2 + 8x - 84 = 0$ by completing the square.

First we will move the 84 to the other side: $x^2 + 8x = 84$. Completing the square is done with the middle coefficient. We need to take 8, divide by 2 and then square the result. Here are the two steps: 1.) $\frac{8}{2} = 4$ 2.) $(4)^2 = 16$. Now we add 16 to both sides of the equation: $x^2 + 8x + 16 = 84 + 16$. From here we can factor the right side: $(x + 4)^2 = 100$. Now take the square root of both sides: $\sqrt{(x + 4)^2} = \pm\sqrt{100}$. This simplifies to: $x + 4 = \pm 10$. Subtract 4 to both sides to get: $x = -4 \pm 10$. This answer is not simplified enough, so do these separately: $x = -4 + 10 = 6$ and $x = -4 - 10 = -14$. Therefore our answers are 6 and -14.

Quadratic Formula

This last method will work for anything in this form: $0 = ax^2 + bx + c$. The formula is: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

You usually want to use this method when it is too hard to factor or if it doesn't factor at all. So a, b, and c in the formula come from the standard form $0 = ax^2 + bx + c$.

EXAMPLE: Solve by using the quadratic formula: $0 = x^2 + 4x + 2$

This one doesn't factor, so the quadratic formula is the only way to solve it. Here $a = 1$, $b = 4$, and $c = 2$. We will put these into the quadratic formula and simplify:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4^2 - 4(1)(2)}}{2(1)} = \frac{-4 \pm \sqrt{16 - 8}}{2} = \frac{-4 \pm \sqrt{8}}{2} = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$$

Notice we broke down the $\sqrt{8}$ by writing $\sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$. Then we divided everything by 2.

EXAMPLE: Solve by using the quadratic formula: $0 = 4x^2 - x + 2$

Here we have $a = 4$, $b = -1$, $c = 2$. Put it into the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(2)}}{2(4)} = \frac{1 \pm \sqrt{1 - 32}}{8} = \frac{1 \pm \sqrt{-31}}{8}$$

We can't take the square root of a negative number, so the answer is undefined in this section, however in the next section we will discuss imaginary numbers, so we will be able to write our answer in a different way instead of saying "undefined".